

Application of the Log-Normal Theory of Mortality to the Valuation of Life Insurance and Annuity Functions

Introduction, Outline, and Discussion

This document discusses and further develops the log-normal theory of mortality of my honours thesis with regard to applications to life insurance calculations. The formulae for the commutation, annuity and insurance valuation functions are stated and described briefly. The functions are then applied to actual valuations, with parameters based on matching the coordinates of the mode of the life table (as suggested by my honours thesis), and compared with the calculations based directly on the commutation functions of the table. It is found that the valuations result in very near correspondence between the two methods, whereby the log-normal theory approximates the table. The calculations cover each of the standard actuarial functions for level payments based on mortality and interest for (1) the log-normal mortality model and compared with those of (2) the life table. Examples of each of these functions are compared based on interest at 4%.

The paper calculates the higher moments of the insurance, annuity, premium, and reserve functions and analyses some of the statistical properties that describe how the random variables underlying these functions behave with regard to the distribution of possible outcomes. These distributions are double log-normal and exhibit skew, heavy kurtosis, and non-normal moments of higher order which is pertinent to decision making based upon these functions; pertinent over and above simply pricing by the mean, with regard to conservatism and likelihood.

Exploiting the fact that the log-normal mortality model is an everywhere continuous random variable amenable to statistical manipulation, and the fact that the insurance functions are stochastic only with regard to this random variable, the appropriate transformations are defined and then used to derive the exact distributions of the insurance, annuity, premium, and reserve functions by the method of transformation by distribution functions in conjunction with the smooth and continuous log-normal mortality curve. The actuarial functions are restricted to those that are in force continuously, which is the exact representation of continuous lifetime (rather than the curtate functions of the life table), and for those that apply whenever the life is still alive, to simplify the analysis. It is felt that the amount of material omitted by concentrating on whole of life functions is not material for the size of this document given that the slightly more complicated functions pertaining to deferment and limited term can be obtained by straight forward manipulations of the whole of life functions and future lifetime variable.

As the transformation method in conjunction with the log-normal theory of mortality results in fully continuous distributions for the life contingent functions and cash flows, equations of density functions are used to define percentiles and derive confidence intervals which are further used to comment on how the use of expected values for these functions is an acceptable statistical measure to summarise the expected experience in the presence of skew and heavy kurtosis for insurance applications. It is determined when the means are, and when they are not, conservative as a means of assigning a price. The expected values, as the actuarial functions used for valuation, are also assessed with regard to median and modal values. This will give industry practitioners (and students) scope for greater understanding of the loss potential and the nature of the possible outcomes of life contingent cash flows including, for example, why longevity risk can pose a large risk to writers of annuities and the providers of pension schemes if they price by the mean since the annuity functions are right skewed and it is more probable than not that the mean value will underprice the actual outcome (the insurance functions are left skewed consistent with the lifetime variable, the annuity functions are reflected versions of the insurance functions). Hence, these functions are not conservative pricing with regard to the provision of annuities [as apposed to receiving premium income, for which they are].

This document presents a simulation algorithm for simulating future lifetimes based upon log-normal mortality and the smooth and continuous distribution, which allows for simulations for demographic purposes as well as for purposes of modelling an insurer's balance sheet for all in-force insurance and annuity business. This provides a means of simulating actual vs expected experience and analysis of profit without having to wait for the experience to emerge. The algorithms are in algebraic form and an example simulation exercise regards 5 lives each aged 40, for life

annuities and insurance, and then a more realistic simulation regarding whole life insurance for 5 policies with various sums insured and ages of policyholders.

Throughout this paper, it should be borne in mind that the working idea of the log-normal theory is, with mortality calculations, if you subtract age from the radix (upper limit to lifespan) and take logs, then all calculations are then based upon a normal bell for which all statisticians (including demographers) and actuaries are well familiar. As this document demonstrates, the log-normal theory is an ideal complement to traditional and well established actuarial theory of life-insurance, the basis derived from that of sound statistical principles combined with the theory of compound interest.

The calculations of this document confirm the goodness of the log-normal theory of mortality of my honours thesis¹ and strongly suggest that this is the law that governs lifespan determined by naturally occurring biological mortality. The log-normal theory of mortality provides both an alternative and a complement to commutation functions and a smooth differentiable mortality curve that can be seen as a ready graduated mortality signal. The log-normal theory can, from this perspective, be viewed as a method for graduating raw mortality data - a graduation based on an underlying theory, rather than a purely mathematical curve fit, as the curve for which generates the crude mortality data. Given the correctness of log-normal mortality by the life-force function of age, a life-table may be seen as an approximation to the true underlying log-normal mortality function, the life table histogram approximating the smooth log-normal curve. It is hypothesized that the log-normal theory of mortality is representative of the lifespan function for biological mortality of all living creatures (all species, not just humans, ... perhaps even the lifecycle of our very own cells) due to the link, via the life-force function, with the Central Limit Theorem for which all of the natural world depends (as much, if not all, of my work does in some shape or form). The log-normal theory provides a convenient, statistically correct, and accurate method of adjusting mortality for rating factors from a base table since such an adjustment to mortality requires only adjustment of the two (three including the radix) mortality parameters, in effect, a simple modification to the life-force function (lighter mortality means stronger life-force and visa-versa). The log-normal model and theory provides a fully continuous and differentiable mortality schedule with calculations at non-integral ages posing little or no additional difficulties and no requirement for any further approximating assumptions such as interpolation methods between integral years of age. Hence, the log-normal theory requires only the best selection of the three reflected log-normal mortality parameters, inclusive of the radix. My honours thesis deemed the best selection of the mortality parameters as those obtained by matching the mode of the log-normal curve to that of the experience, in this case, the Human Mortality Database life table for 2004 Australian males (the p_x, μ_x from the q_x).

It is recommended that anyone doing the mathematics of this document for themselves do so with a programmable calculator that can integrate. This is how this mathematics was calculated for the log-normal curve, with the density, cumulative density, percentiles, annuity and insurance functions programmed into the calculator with numerical integrations for the latter functions (and also to determine the mortality parameters [standard deviation of life-force] from the table included as an appendix). The alternative is MATLAB or Mathematica. MATLAB is how the functions were graphed.

The log-normal theory of mortality is an excellent pedagogical tool regardless of whether or not it is appropriate for actual pricing of insurance and annuities. A normal curve presents a good curve for mortality from non-natural causes, so for actual pricing, the log normal theory should be utilized in conjunction with a normal curve for non-biological mortality [as a limiting CLT sum of Poisson variables for such factors as road accidents] which is distinct from that of natural causes relating to the aging process. This will also provide for some interesting statistics and demographics.

Reflected Log-normal Actuarial Functions

The functions pertaining to the log-normal theory of mortality are original to my honours thesis and this paper, which was first written in 2008 (in the first year of my Finance PhD), immediately after completing my honours thesis (submitted in person to Shauna Ferris in 2007 at Macquarie University) for which my research developed, among other things, the log-normal theory of mortality (including the pdf, cdf, moment

¹ Lloyd-Jones, A., (2007) Actuarial Honours Thesis. Department of Actuarial Studies, Macquarie University.

matched parameters, force of mortality, and demographic functions) by comparing all eligible density functions for mortality. The insurance, annuity, and commutation functions (and simulation algorithm) for the log-normal theory are from, and original to, this paper (rather than the honours thesis);

0.1. Probability density

$$f_{T_0}(x) = {}_x p_0 \mu_x = \frac{1}{(\omega - x)\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log(\omega - x) - \mu)^2}{2\sigma^2}\right)$$

or, in terms of a standard unit normal density,

$$f_{T_0}(x) = {}_x p_0 \mu_x = \frac{1}{(\omega - x)\sigma} \phi\left(\frac{\log(\omega - x) - \mu}{\sigma}\right)$$

0.2. Survival distribution

$$S_{T_0}(x) = {}_x p_0 = \Phi\left(\frac{\log(\omega - x) - \mu}{\sigma}\right)$$

0.3. Cumulative distribution

$$F_{T_0}(x) = {}_x q_0 = \Phi\left(\frac{\mu - \log(\omega - x)}{\sigma}\right)$$

0.4. Force of mortality

$$\lambda_{T_0}(x) = \mu_x = f_{T_0}(x) / S_{T_0}(x) = \frac{\phi([\log(\omega - x) - \mu] / \sigma)}{\sigma(\omega - x)\Phi([\log(\omega - x) - \mu] / \sigma)}$$

0.5. Reflected log-normal parameters (matched mode)

The standard deviation of life-force σ is obtained from the equation

$$\frac{\exp(-0.5\sigma^2)}{\sigma} = \sqrt{2\pi} (\omega - \text{Mode}(T_0)) f_{T_0}(\text{Mode}(T_0))$$

which is solved numerically; the mean of life-force μ is obtained by

$$\mu = \log(\omega - \text{Mode}(T_0)) + \sigma^2$$

0.6. Conditional probability density

$$f_{T_x}(y) = {}_y p_x \mu_{x+y} = \frac{f_{T_0}(x+y)}{S_{T_0}(x)} = \frac{\phi([\log(\omega - x - y) - \mu] / \sigma)}{\sigma(\omega - x - y)\Phi([\log(\omega - x) - \mu] / \sigma)}$$

0.7. Conditional survival distribution

$$S_{T_x}(y) = {}_y p_x = \frac{S_{T_0}(x+y)}{S_{T_0}(x)} = \frac{\Phi([\log(\omega - x - y) - \mu] / \sigma)}{\Phi([\log(\omega - x) - \mu] / \sigma)}$$

0.8. Conditional cumulative distribution

$$\begin{aligned} F_{T_x}(y) = {}_y q_x &= \frac{F_{T_0}(x+y) - F_{T_0}(x)}{S_{T_0}(x)} = \frac{S_{T_0}(x) - S_{T_0}(x+y)}{S_{T_0}(x)} \\ &= \frac{\Phi([\ln(\omega - x) - \mu] / \sigma) - \Phi([\ln(\omega - x - y) - \mu] / \sigma)}{\Phi([\ln(\omega - x) - \mu] / \sigma)} \end{aligned}$$

0.9. Conditional force of mortality

$$\lambda_{T_x}(y) = \lambda_{T_0}(x+y)$$

0.10. Demographic statistics

0.10.1. Expectation of life (Mean of future lifetime)

$$\text{Mean}(T_x) = \omega - x - \exp\left(\mu + \frac{1}{2}\sigma^2\right) \frac{\Phi([\log(\omega - x) - \mu] / \sigma - \sigma)}{\Phi([\log(\omega - x) - \mu] / \sigma)}$$

0.10.2. Median of future lifetime

$$\text{Median}(T_x) = \omega - x - \exp\left(\mu + \sigma\Phi^{-1}(0.50S_{T_0}(x))\right)$$

0.10.3. Mode of future lifetime

$$\text{Mode}(T_x) = \omega - \exp\left(\mu - \sigma^2\right)$$

for any age $x < \text{Mode}(T_0)$. $\text{Mode}(T_x) = x$ thereafter.

0.11. Continuous commutation functions: $D_x, \bar{N}_x, \bar{S}_x, \bar{C}_x, \bar{M}_x, \bar{R}_x$

0.11.1. Survival commutation functions $D_x, \bar{N}_x, \bar{S}_x$

$$D_x = l_0 \times v^x \Phi((\ln(\omega - x) - \mu) / \sigma) d$$

$$\bar{N}_x = l_0 \times \int_0^{\omega-x} v^{x+t} \Phi((\ln(\omega - x - t) - \mu) / \sigma) dt$$

$$\bar{S}_x = l_0 \times \int_0^{\omega-x} \int_0^y v^{x+t} \Phi((\ln(\omega - x - t) - \mu) / \sigma) dt dy$$

0.11.2. Mortality commutation functions $\bar{C}_x, \bar{M}_x, \bar{R}_x$

$$\bar{C}_x = l_0 \times \int_0^1 v^{x+t} \frac{\phi((\ln(\omega - x - t) - \mu) / \sigma)}{\sigma(\omega - x - t)} dt$$

$$\bar{M}_x = l_0 \times \int_0^{\omega-x} v^{x+t} \frac{\phi((\ln(\omega - x - t) - \mu) / \sigma)}{\sigma(\omega - x - t)} dt$$

$$\bar{R}_x = l_0 \times \int_0^{\omega-x} \int_0^y v^{x+t} \frac{\phi((\ln(\omega - x - t) - \mu) / \sigma)}{\sigma(\omega - x - t)} dt dy$$

0.12. annuity functions $\bar{a}_x, {}_m\bar{a}_x, \bar{a}_{x:\overline{n}|}, {}_m\bar{a}_{x:\overline{n}|}$

Probability density function of a

$$f_a(y) = \frac{1}{1 - \delta y} \frac{\Phi([\mu - \log(\omega - x + (1/\delta)\log(1 - \delta y))] / \sigma)}{\sigma(\omega - x + (1/\delta)\log(1 - \delta y))\Phi([\log(\omega - x) - \mu] / \sigma)}$$

$$F_a(y) = \frac{\Phi([\mu - \log(\omega - x + (1/\delta)\log(1 - \delta y))] / \sigma)}{\Phi([\log(\omega - x) - \mu] / \sigma)} \quad y \in (0, \bar{a}_{\omega-x})$$

$$\bar{a}_x = \int_0^\infty v^t \frac{\Phi([\log(\omega - x - t) - \mu] / \sigma)}{\Phi([\log(\omega - x) - \mu] / \sigma)} dt$$

$$\ddot{a}_x = \bar{a}_x + 1/2 + (\mu_x + \delta)/12$$

0.13. Assurance functions ${}_n E_x, \bar{A}_x, \bar{A}_{x:\overline{n}|}, \bar{A}_{x:\overline{n}|}$

Probability density function of A

$$f_A(y) = \frac{1}{\delta y} \frac{\phi([\log(\omega - x + (1/\delta)\log y) - \mu] / \sigma)}{\sigma(\omega - x + (1/\delta)\log y)\Phi([\log(\omega - x) - \mu] / \sigma)} \quad y \in (v^{\omega-x}, 1)$$

$$F_A(y) = \frac{\Phi([\log(\omega - x + (1/\delta)\log y) - \mu] / \sigma)}{\Phi([\log(\omega - x) - \mu] / \sigma)} \quad y \in (v^{\omega-x}, 1)$$

$$A_x = 1 - d\ddot{a}_x \quad \bar{A}_x = 1 - \delta\bar{a}_x$$

0.14. Premium ratio

$$f_P(p) = \frac{1}{p(p + \delta)} \frac{\phi([\log(\omega - x - (1/\delta)\log(1 + \delta/p)) - \mu] / \sigma)}{\sigma(\omega - x - (1/\delta)\log(1 + \delta/p))\Phi([\log(\omega - x) - \mu] / \sigma)}$$

$$F_P(p) = \frac{\Phi([\log(\omega - x - (1/\delta)\log(1 + \delta/p)) - \mu] / \sigma)}{\Phi([\log(\omega - x) - \mu] / \sigma)} \quad p \in \left(\frac{\delta v^{\omega-x}}{1 - v^{\omega-x}}, \infty\right)$$

0.15. Reserve function

$$f_V(y) = \frac{1}{P + \delta y} \frac{\phi\left(\left(\log\left(\omega - x - t - \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) - \mu\right) / \sigma\right)}{\sigma \left(\omega - x - t - \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \Phi\left((\log(\omega - x - t) - \mu) / \sigma\right)}$$

$$F_V(y) = \frac{\Phi\left(\left(\log\left(\omega - x - t - \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) - \mu\right) / \sigma\right)}{\Phi\left((\log(\omega - x - t) - \mu) / \sigma\right)}$$

$$y \in \left((1 + P/\delta)v^{\omega - x - t} - P/\delta, 1\right)$$

0.16. Simulation algorithm

$$T_x = \omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(u \Phi\left(\frac{\log(\omega - x) - \mu}{\sigma}\right)\right)\right)$$

where u is a simulated variate from a unit uniform variable $U \sim U(0,1)$

Part 1: Comparisons with life-table

0. The mortality parameters of a log-normal lifetime function

I do not yet have an entirely satisfactory method of pinning down the exact mode and modal frequency of a life-table from within sampling variation in the d_x and, depending on what values are selected for the mode and modal frequency, there is some discretionary judgement required for a good selection of the terminal age ω which then determines the mean and standard deviation of the life-force function by reference to the mode; there is still some scope for improvement in the parameters that are utilized in this document. Improvement in the parameterization would probably be best viewed in conjunction with a spliced normal [from heterogeneous Poisson CLT] for non-biological mortality.

The mode of the life-table, by interpolation (originally by a spline), is 85.6187 years, which has modal frequency 0.03833774 (somewhat of an educated guesstimate). The radix of the HMD life-table is 110+ years of age. The continuous log-normal mortality schedule is therefore defined to the end of the 110th year of age, i.e., to 111. There is some additional years for the grouping (the table has grouped all deaths above 110 as 110); I have utilized 112 for omega

$$\omega = 112$$

It may well be best to take omega as being larger than this, say in the vicinity of 120. The equation for the standard deviation of life-force σ , by matching the coordinates of the mode to the HMD life-table is, from my honours thesis²,

$$\frac{\exp(-0.5\sigma^2)}{\sigma} = \sqrt{2\pi} (\omega - \text{Mode}(T_0)) f_{T_0}(\text{Mode}(T_0))$$

and for the mean of life-force μ

$$\mu = \log(\omega - \text{Mode}(T_0)) + \sigma^2$$

Solving the first equation numerically and the second by substitution

$$\frac{\exp(-0.5\sigma^2)}{\sigma} = \sqrt{2\pi} (112 - 85.6187) \times 0.03833774 \Rightarrow \sigma = 0.368547$$

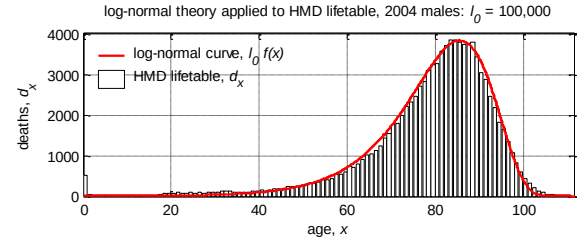
$$\mu = \log(112 - 85.6187) + 0.368547^2 = 3.408482$$

which results in the mortality parameters;

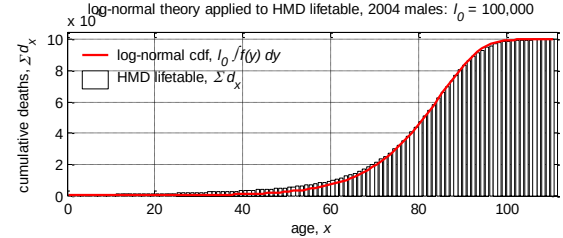
$$\mu = 3.408482 \text{ log-years (mean of life-force)}$$

$$\sigma = 0.368547 \text{ log-years (standard deviation of life-force)}$$

These mortality parameters generate the curve over the life-table histogram from HMD 2004 males in the following diagram



The cumulative reflected log-normal density function vs the life-table ogive;



Interest is assumed flat at 4% per annum. The elements of the life table to be compared are, as excerpts from the complete tables in the appendix,

Life table entries (q_x are from HMD, l_x and d_x are rounded)

[x]	l_x	d_x	p_x	q_x	e_x
20	98980	83	0.99916	0.00084	58.81
30	98108	109	0.99889	0.00111	49.29
40	96973	144	0.99852	0.00148	39.82
50	95011	285	0.99700	0.00300	30.53
60	90788	730	0.99196	0.00804	21.71
70	80256	1754	0.97814	0.02186	13.88
80	55888	3255	0.94176	0.05824	7.71
90	19156	3047	0.84095	0.15905	3.79

where the d_x and l_x are rounded.

Survival commutation functions

[x]	D_x	N_x	S_x
20	45173.07163	1039674.946	20384683.03
30	30248.60453	659974.4104	11821795.86
40	20198.45434	406034.5754	6449089.207
50	13369.24556	236886.6558	3207162.859
60	8630.305717	125878.6608	1377484.758
70	5153.98862	56010.54836	462069.8034
80	2424.651321	17300.35005	98924.94784
90	561.4283234	2399.404994	8962.85432

Mortality commutation functions

[x]	C_x	M_x	R_x
20	36.48594247	5185.588629	255650.032
30	32.2845683	4864.988277	205291.162
40	28.74395425	4581.754808	157993.742
50	38.56513143	4258.235252	113535.148
60	66.71890189	3788.833675	72899.2383
70	108.3328762	2999.751672	38239.2442
80	135.780474	1759.268154	13496.0066
90	85.86074503	469.1584281	2054.99298

² Lloyd-Jones, A., (2007) Actuarial Honours Thesis. Department of Actuarial Studies, Macquarie University.

1.1. ANNUITIES, a : Evaluation of annuity functions, payable continuously, by the log-normal theory and the lifetable.

1.1.1. Continuous whole life annuity $\bar{a}_x$ – a benefit of 1 per annum, payable continuously to the annuitant (x), while the annuitant is alive, starting immediately: *whenever the life is alive, pay him 1*

$$\begin{aligned}\bar{a}_x &= \int_0^\infty \bar{a}_{x|t} p_x \mu_{x+t} dt \\ &= \int_0^\infty v^t p_x dt \\ &= \int_0^\infty v^t S_{T_x}(t) dt \\ &= \frac{1}{k} \int_0^\infty v^t \Phi\left(\frac{\log(\omega - x - t) - \mu}{\sigma}\right) dt\end{aligned}$$

by integration by parts, and where

$$k = S_{T_0}(x) = \Phi\left(\frac{\log(\omega - x) - \mu}{\sigma}\right)$$

because of symmetry about the origin of the standard normal. The integral must be evaluated numerically [for example on a Ti89]. Say a life now aged 40 purchases a life annuity with money worth 4%. The purchase price of this annuity according to log-normal mortality with modal matched parameters is

$$k = \Phi\left(\frac{\log(112 - 40) - 3.4085}{0.3685}\right) = \Phi(2.3557) = 0.99076$$

$$\begin{aligned}\bar{a}_{40} &= \frac{1}{k} \int_0^{72} 1.04^{-t} \Phi\left(\frac{\log(112 - 40 - t) - 3.4085}{0.3685}\right) dt \\ &= 19.4077 / 0.99076 \\ &= 19.5888\end{aligned}$$

Valuing this annuity using commutation functions at 4% in the appendix for the life table that the modal matched reflected log-normal model is derived (the life table, on which the D_x , N_x , and S_x are based, is also in the appendix) is

$$\begin{aligned}\bar{a}_{40} &= N_{40} / D_{40} \\ &= 406034.6 / 20198.45 \\ &= 20.1023\end{aligned}$$

Correcting for continuous payments by assuming a uniform distribution over each year of life in the table, and hence deducting $\frac{1}{2}$ from the annuity, gives

$$\bar{a}_{40} = 19.6023$$

Improving the correction using the more sophisticated formula³

$$\begin{aligned}\bar{a}_x &= \bar{a}_x - 1/2 - (\mu_x + \delta)/12 \\ &\equiv \bar{a}_x - 1/2 - (q_x + \delta)/12\end{aligned}$$

gives

$$\begin{aligned}\bar{a}_{40} &= \bar{a}_{40} - 1/2 - (q_{40} + \delta_{4\%})/12 \\ &\equiv 20.1023 - 1/2 - (0.00148 + 0.03922)/12 \\ &= 19.5989\end{aligned}$$

This premium is almost identical to that priced by the log-normal model; the premium for the whole of life annuity for a life now aged forty $\bar{a}_{40}$ is \$19.5888 for the log-normal curve and \$19.5989 from the table for which the log-normal curve is approximating (or smoothing).

The valuation from the original table is very close to that of the log-normal model, a difference of only one cent. The log-normal model has priced this annuity with an error of approximately 0.05%. Depending on the actuaries circumstances, and seeing that the correction for continuous payments is an approximation, this error is probably not material in the majority of cases, especially given today's uncertain investment environment and the unlikelihood of a flat yield curve. This percentage error

varies according to the age at which the annuity is valued, although the error of 0.05% is indicative of the general pricing difference.

Note that the scaling factor k of 0.99076 is the reflected log-normal value of

$$S_{T_0}(40) = I_{40} / I_0$$

The life table in appendix A evaluates this probability as $I_{40} / I_0 = 0.96973$. The log-normal model probability is in error for this survival probability as the lifetable includes all sources of mortality while the log-normal curve applies exclusively to biological mortality; the error is entirely due to an accumulation of differences in the left tail. This is the only significant divergence between the table and the log-normal curve with matched coordinates of the most frequent age of death. Considering that the model has only two parameters (not including the radix ω), this is very reasonable and amounts to approximately 2,000 more deaths per 100,000 radix in the table in the early years of life because of the fact that the log-normal theory applies to biological mortality. These lives are the result of infant mortality, including 530 deaths in the first year of life, and the accident hump (such as road fatalities, suicides, occupational mortality, etc). Infant mortality is not relevant to insurance and annuity valuations. A more sophisticated model of the lifespan variable T_0 would mix the log-normal theory for biological mortality with an exponential distribution for infant mortality and a normal curve for death from un-natural causes predominant to young adult life. As medical science and society progresses, non-biological mortality will become less and less prevalent as it has experienced over the recent past⁴. This may be seen as a demographic indicator of the goodness of a society or population. For all practical purposes of insurance, infant mortality could reasonably be zeroed (zero/left-tail modified) and the lifetime variable be made almost near perfectly accurate (for all ages) with the mixture⁵ of

- (1) a normal curve for non-natural mortality (additive, for example, a CLT from the supposition of many heterogeneous Poisson variables);
- (2) a reflected log-normal curve for naturally occurring mortality (non-additive, a CLT on aggregations of the life-force function).

x	annuity payable continuously			
	$\bar{a}_x$ @ 4%		Absolute	Relative
	table	log-normal	difference	difference
20	22.5120313	22.6977	-0.186	-0.82%
30	21.3149815	21.4089	-0.094	-0.44%
40	19.598868	19.5888	0.010	0.05%
50	17.2152567	17.1115	0.104	0.60%
60	14.0817165	13.9438	0.138	0.98%
70	10.3623279	10.2707	0.092	0.88%
80	6.62706918	6.5614	0.066	0.99%
90	3.75722868	3.413	0.344	9.16%

The dollar difference between the premium for continuous payments by the log-normal curve and that by the life table is less \$1 for every age. This is less than 1% in all cases except age 65, which it is only just more than 1%, and the last ages of the table.

Rather than modifying the annuity due of the table to a continuous annuity for comparison with the continuous annuity of the log-normal curve, one may instead modify the continuous log-normal annuity to a life annuity with payments in advance by the formula

⁴ Pollard, J. H. (1995), On the Changing Shape of the Australian Mortality Curve. *Actuarial Studies and Demographic Research Paper Series*. Macquarie University.

⁵ mixtures of normals date back almost to Gauss himself. For example

Everitt, B. S., (2002), *The Cambridge Dictionary of Statistics*, 2nd Edition. Cambridge University Press;

quote an example from 1894 by Karl Pearson. For mixing a normal curve into a reflected log-normal curve for mortality, I will be referring to

Klugman, S. A., Panjer, H. H., Willmot, G. E., (1998), *Loss Models*. New York: Wiley;

and for anything pertaining to zero-modifications (since this is where I studied zero-modification in the sound statistical context of their book).

³ Neill, A. (1979), *Life Contingencies*. Heineman Publishing. London.

$$\ddot{a}_x = \bar{a}_x + 1/2 + (\mu_x + \delta)/12$$

This results in the following table for lifetime annuities of a benefit stream of 1 per annum paid in advance;

annuity payable in advance				
	$\ddot{a}_x @ 4\%$		Absolute	Relative
x	table	log-normal	difference	difference
20	23.0153697	23.201	-0.186	-0.81%
30	21.8183424	21.9122	-0.094	-0.43%
40	20.1022597	20.0921	0.010	0.05%
50	17.7187751	17.615	0.104	0.59%
60	14.5856549	14.4477	0.138	0.95%
70	10.8674179	10.7758	0.092	0.84%
80	7.13519091	7.06977	0.065	0.92%
90	4.27375124	3.93036	0.343	8.03%

for which the correspondence is similarly close between the curve by the log-normal theory and that of the lifetable.

1.1.2. Deferred continuous whole life annuity ${}_m\bar{a}_x$ – a benefit of 1 per annum, payable continuously to the annuitant (x), while the annuitant is alive, starting in m years time: *whenever the life is alive after m years from now, pay him 1*

$$\begin{aligned} {}_m\bar{a}_x &= v^m {}_mP_x \bar{a}_{x+m} \\ &= {}_mE_x \bar{a}_{x+m} \end{aligned}$$

where ${}_mE_x$ is a pure endowment for a life aged x payable on survival in m periods time, and acts as the a m year discount factor for interest and mortality for a life aged x .

Say a life now aged 40 purchases a lifetime annuity with continuous payments of 1 per annum that begin in 10 years time. The actuarial present value is

$${}_{10}\bar{a}_{40}$$

Valuing the annuity by the log-normal model derived from the life table in appendix A; the value of the 10 year pure endowment for a life aged 40 as the 10 year discount factor for interest and mortality, is

$$\begin{aligned} {}_{10}E_{40} &= v_{4\%}^{10} \Phi\left(\frac{\log(112-50)-3.4085}{0.3685}\right) \Bigg/ \Phi\left(\frac{\log(112-40)-3.4085}{0.3685}\right) \\ &= v_{4\%}^{10} \Phi(1.9500) / \Phi(2.3557) \\ &= 0.6756 \times 0.9744 / 0.9908 \\ &= 0.66442 \end{aligned}$$

The value of the immediate annuity for a life aged 50 is

$$\begin{aligned} \bar{a}_{50} &= \frac{1}{k} \int_0^{62} 1.04^{-t} \Phi\left(\frac{\log(112-50-t)-3.4085}{0.3685}\right) dt \\ &= 16.6736 / 0.9744 \\ &= 17.1115 \end{aligned}$$

$$\text{where } k = \Phi\left(\frac{\log(112-50)-3.4085}{0.3685}\right) = \Phi(1.9500) = 0.9744$$

The value of the 10 year deferred annuity for a life aged 40 is then

$$\begin{aligned} {}_{10}\bar{a}_{40} &= {}_{10}E_{40} \bar{a}_{50} \\ &= 0.66442 \times 17.1115 \\ &= 11.3692 \end{aligned}$$

Valuing the annuity with the commutation functions of the life table in the appendix; the value of the pure endowment is

$$\begin{aligned} {}_{10}E_{40} &= N_{50} / D_{40} \\ &= 13369.25 / 20198.45 \\ &= 0.661895 \end{aligned}$$

The value of the continuous life annuity for a life aged 50 is

$$\begin{aligned} \bar{a}_{50} &= N_{50} / D_{40} - 1/2 - (q_{50} + \delta)/12 \\ &= 23688.67 / 13369.25 - 1/2 - (.00300 + \delta_{4\%})/12 \\ &= 17.2153 \end{aligned}$$

and the value of the 10 year deferred life annuity is

$$\begin{aligned} {}_{10}\bar{a}_{40} &= 0.661895 \times 17.2152567 \\ &= 11.3947 \end{aligned}$$

The premium for a 10 year deferred annuity for a life aged forty ${}_{10}\bar{a}_{40}$ is \$11.37 for the log-normal curve and \$11.40 for the lifetable. The log-normal model with matched modal coordinates has underpriced the annuity based directly on the table by slightly less than 3 cents per dollar benefit, a proportional error of 0.25%. Again, this is very reasonable in light of the fact that there is a fifty fold reduction in the degrees of freedom upon which the valuation model is based.

1.1.3. Continuous temporary life annuity $\bar{a}_{x:n|}$ – a benefit of 1 per annum, payable continuously to the annuitant (x), while the annuitant is alive up to a maximum of n years worth of payments, starting immediately: *whenever the life is alive, pay him 1. Do this for a maximum of n years*

$$\begin{aligned} \bar{a}_{x:n|} &= \bar{a}_x - v^n {}_nP_x \bar{a}_{x+n} \\ &= \bar{a}_x - {}_nE_x \bar{a}_{x+n} \\ &= \bar{a}_x - {}_n\bar{a}_x \end{aligned}$$

Each of the functions on the right-hand-side of the equation where just defined according to the log-normal mortality model. Say a life now aged 40 purchases a stream of payments, assumed paid continuously, of 1 per annum starting today for a maximum of 20 years with money worth 4% per annum compound. The actuarial present value of this annuity is

$$\begin{aligned} \bar{a}_{40:20|} &= \bar{a}_{40} - v_{4\%}^{20} {}_{20}P_{40} \bar{a}_{60} \\ &= \bar{a}_{40} - {}_{20}E_{40} \bar{a}_{60} \\ &= \bar{a}_{40} - {}_{20}\bar{a}_{40} \end{aligned}$$

Evaluating the functions on the right hand side,

$$\bar{a}_{40} = 19.5888$$

$$\begin{aligned} {}_{20}E_{40} &= v_{4\%}^{20} \Phi\left(\frac{\log(112-60)-3.4085}{0.3685}\right) \Bigg/ \Phi\left(\frac{\log(112-40)-3.4085}{0.3685}\right) \\ &= v_{4\%}^{20} \Phi(1.4727) / \Phi(2.3557) \\ &= 0.4564 \times 0.9296 / 0.9908 \\ &= 0.42821 \end{aligned}$$

$$\bar{a}_{60} = 13.9438$$

Combining the components for the log-normal value of the annuity

$$\begin{aligned} \bar{a}_{40:20|} &= 19.5888 - 0.42821 \times 13.9438 \\ &= 13.6179 \end{aligned}$$

Valuing the 20 year term life annuity by the commutation functions gives

$$\begin{aligned} \bar{a}_{40:20|} &= \ddot{a}_{40} - {}_{20}E_{40} \ddot{a}_{60} \\ &= (N_{40} - N_{60}) / D_{40} \\ &= 13.87017 \end{aligned}$$

and adjusting for continuous payments

$$\begin{aligned} \bar{a}_{40:20|} &= \bar{a}_{40} - {}_{20}E_{40} \bar{a}_{60} \\ &= (N_{40} / D_{40} - 1/2) - (N_{60} / D_{40} - 1/2) D_{60} / D_{40} \\ &= (406034.6 / 20198.45 - 0.5) \\ &\quad - (125878.7 / 8630.31 - 0.5) \times 8630.31 / 20198.45 \\ &= 13.5838 \end{aligned}$$

or with the slightly more accurate formulae

$$\begin{aligned}\bar{a}_{40:\overline{20}|} &= \bar{a}_{40} - {}_{20}E_{40} \bar{a}_{60} \\ &= (N_{40}/D_{40} - 1/2 - q_{40}/12 - \delta/12) \\ &\quad - (N_{60}/D_{40} - 1/2 - q_{60}/12 - \delta/12) D_{60}/D_{40} \\ &= 19.59886802 - 14.0817165 \times 8630.31/20198.45 \\ &= 13.5821\end{aligned}$$

The temporary annuity premium $\bar{a}_{40:\overline{20}|}$ based directly on the modal-coordinate-matched log-normal model of \$13.62 differs from that based directly on the life table of \$13.58 by only 4 cents, a proportional error of only 0.25%, a quarter of 1 percent. This confirms the goodness of the log-normal theory of mortality and the associated concept of life-force. Actuarial valuations may now rest on a conceptual theory that links finance to nature and its fundamental concept of variation and large numbers; the Central Limit Theorem. The error that is present is almost entirely due to non-biological mortality from the infant years of life and the years of early adulthood and which is near negligible for the majority of actuarial calculations for insurance and annuities.

1.1.4. Deferred continuous temporary life annuity ${}_m\bar{a}_{x:n|}$ – a benefit of 1 per annum, payable continuously to the annuitant (x), while the annuitant is alive up to a maximum of n years worth of payments, starting in m years time: *whenever the life is alive after m years from now, pay him 1. Do this for a maximum of n years*

$$\begin{aligned}{}_m\bar{a}_{x:n|} &= v^m {}_mP_x \bar{a}_{x+m:n|} \\ &= {}_mE_x \bar{a}_{x+m:n|} \\ &= v^m {}_mP_x (\bar{a}_{x+m} - v^n {}_nP_{x+m} \bar{a}_{x+m+n}) \\ &= {}_mE_x (\bar{a}_{x+m} - {}_nE_{x+m} \bar{a}_{x+m+n})\end{aligned}$$

Say a life now aged 40 exact now purchases an annuity with continuous payments of 1 per annum that will begin in 10 years and last for 20. The actuarial present value of this stream of payments is

$${}_{10}\bar{a}_{40:\overline{20}|} = {}_{10}E_{40} (\bar{a}_{50} - {}_{20}E_{50} \bar{a}_{70})$$

Evaluating the components of the right-hand-side separately according to log-normal mortality;

(1) Life contingent present value factors;

$${}_{10}E_{40} = 0.66442$$

$$\begin{aligned}{}_{20}E_{50} &= v_{4\%}^{20} \Phi\left(\frac{\log(112-70)-3.4085}{0.3685}\right) / \Phi\left(\frac{\log(112-50)-3.4085}{0.3685}\right) \\ &= v_{4\%}^{20} \Phi(0.8932) / \Phi(1.9500) \\ &= 0.4564 \times 0.8141 / 0.9744 \\ &= 0.3813\end{aligned}$$

(2) Annuity factors

$$\bar{a}_{50} = 17.1115$$

$$\bar{a}_{70} = 10.2707$$

(3) Combining the factors for the value of the annuity

$$\begin{aligned}{}_{10}\bar{a}_{40:\overline{20}|} &= 0.66442 \times (17.1115 - 0.3813 \times 10.2707) \\ &= 8.7671\end{aligned}$$

Valuing the actuarial present value using the table;

$$\begin{aligned}{}_m\ddot{a}_{x:n|} &= {}_mE_x (\ddot{a}_{x+m} - {}_nE_{x+m} \ddot{a}_{x+m+n}) \\ &= \frac{D_{x+m}}{D_x} \left(\frac{N_{x+m}}{D_{x+m}} - \frac{D_{x+m+n}}{D_{x+m}} \frac{N_{x+m+n}}{D_{x+m+n}} \right) \\ &= \frac{N_{x+m} - N_{x+m+n}}{D_x}\end{aligned}$$

by the beautiful simplicity of commutation functions and the intuitive interweaving of relative frequency, probability, and compound interest;

$${}_{10}\bar{a}_{40:\overline{20}|} = \frac{N_{50} - N_{70}}{D_{40}} = \frac{2368867 - 560105}{2019845} = 8.9550$$

Adjusting for continuous payments

$$\begin{aligned}{}_m\bar{a}_{x:n|} &\doteq {}_mE_x (\bar{a}_{x+m} - 1/2 - {}_nE_{x+m} (\ddot{a}_{x+m+n} - 1/2)) \\ &= \frac{D_{x+m}}{D_x} \left(\frac{N_{x+m}}{D_{x+m}} - 1/2 - \frac{D_{x+m+n}}{D_{x+m}} \left(\frac{N_{x+m+n}}{D_{x+m+n}} - 1/2 \right) \right)\end{aligned}$$

$$\begin{aligned}{}_{10}\bar{a}_{40:\overline{20}|} &\doteq \frac{D_{50}}{D_{40}} \left(\frac{N_{50}}{D_{50}} - 1/2 - \frac{D_{70}}{D_{50}} \left(\frac{N_{70}}{D_{70}} - 1/2 \right) \right) \\ &= \frac{13369.25}{20198.45} \left(\frac{236886.7}{13369.25} - 1/2 - \frac{5153.99}{13369.25} \left(\frac{56010.5}{5153.99} - 1/2 \right) \right) \\ &= 8.7516\end{aligned}$$

or with the more accurate formulae

$$\begin{aligned}{}_{10}\bar{a}_{40:\overline{20}|} &\doteq \frac{D_{50}}{D_{40}} \left(\frac{N_{50}}{D_{50}} - \frac{1}{2} - \frac{\delta + q_{50}}{12} - \frac{D_{70}}{D_{50}} \left(\frac{N_{70}}{D_{70}} - \frac{1}{2} - \frac{\delta + q_{70}}{12} \right) \right) \\ &= \frac{13369.25}{20198.45} \left(17.21525674 - \frac{5153.99}{13369.25} \times 10.36232787 \right) \\ &= 8.7506\end{aligned}$$

The premium for the temporary 20 year annuity, beginning in 10 years, for a life aged 40, ${}_{10}\bar{a}_{40:\overline{20}|}$, is \$8.75 from the life table and \$8.77 from the reflected log-normal curve. Hence, the log-normal mortality model with modal coordinate matched parameters has over-priced the annuity by about 2 cents for a benefit stream of 1 per annum. This is a relative error of slightly less than 0.2%, one fifth of one percent, per dollar of annual benefit.

1.2. INSURANCE, A: Evaluation of insurance functions, payable immediately upon claim, by the log-normal theory and the life table.

1.2.1. Pure endowment ${}_nE_x$ – a benefit of 1 payable to the policyholder (x) upon survival n years from the inception of the policy: *if the life survives n years, pay him 1*

The pure endowment is a discount factor for interest and mortality, a combined discount factor and probability, used to shift payments backwards or forwards through time; compound interest inclusive of mortality. In addition to the technical definition, the function is also a traditional life insurance policy, similar to a simple bank or savings account. A pure endowment is analogous to the zero coupon bond of finance, with additional consideration of policyholder mortality or survivorship. This function was utilized in the preceding annuity calculations. It is defined here again for consistency of presentation.

$${}_nE_x = v^n {}_nP_x = v^n {}_{x+n}P_0 / {}_xP_0 = v^n S_{T_0}(x+n) / S_{T_0}(x)$$

There is discrepancy for these functions between the table and the reflected log-normal curve due to the log-normal theory of mortality applying only to biological mortality. This discrepancy does not factor largely in the insurance and payment functions but should be borne in mind whenever the log-normal theory of mortality is utilized. For insurance calculations, infant mortality is not material and can be zeroed so as not to skew the difference between the log-normal curve and the crude data of the experience. I was considering zeroing infant mortality but have decided to keep the table as is from values of the Human Mortality Database, for the task of comparing the log-normal theory consistent with the q_x .

1.2.2. Whole of life insurance $\bar{A}_x$ – a benefit of 1 payable to the estate of the policy holder (x) at the end of the policy holders life: *whenever the life dies, pay him 1*

$$\begin{aligned}\bar{A}_x &= \int_0^\infty v^t {}_tP_x \mu_{x+t} dt \\ &= \int_0^\infty v^t \frac{d(-{}_tP_x)}{dt} dt \\ &= -\int_0^\infty v^t d{}_tP_x \\ &= [-v^t {}_tP_x]_0^\infty - \int_0^\infty \delta v^t {}_tP_x dt \\ &= 1 - \delta \bar{a}_x\end{aligned}$$

which links the assurance functions to the annuity functions; in the context here, whole life assurances payable immediately on claim in terms of lifetime annuities with continuous payments. Evaluating a whole of life

assurance issued to a life aged 40 according to the log-normal mortality model;

$$\begin{aligned}\bar{A}_{40} &= 1 - \delta_{4\%} \bar{a}_{40} \\ &= 1 - 19.5888 \times \ln(1.04) \\ &= 0.2317\end{aligned}$$

Evaluating the insurance function using the 4% commutation functions,

$$\begin{aligned}A_x &= M_x / D_x \\ A_{40} &= M_{40} / D_{40} \\ &= 4581.74 / 20198.45 \\ &= 0.22684\end{aligned}$$

Adjusting for immediate payment of benefit upon claim, rather than at the end of the year that the claim is made, by accumulating the beginning of period figure for one half of one year according to the claim acceleration formula⁶

$$\bar{A}_x \approx (1+i)^{0.5} A_x$$

gives

$$\bar{A}_{40} = (1+i)^{0.5} A_{40} = \sqrt{1.04} A_{40} = \sqrt{1.04} \times 0.22684 = 0.2313$$

The premium for a whole of life insurance for a life aged 40, $\bar{A}_{40}$, by the modal matched log-normal cure is \$0.2317 and that based upon the life table from which the log-normal approximates is \$0.2313. These two premiums differ only in the fourth decimal place; the relative error is only 0.17% (less than one fifth of one percent).

x	payable at moment of death			
	$\bar{A}_x @ 4\%$		Absolute	Relative
	table	log-normal	difference	difference
20	0.11706208	0.109779	0.007	6.22%
30	0.16401123	0.160327	0.004	2.25%
40	0.23131842	0.231715	0.000	-0.17%
50	0.32480535	0.328877	-0.004	-1.25%
60	0.44770504	0.453115	-0.005	-1.21%
70	0.59358211	0.597177	-0.004	-0.61%
80	0.74008162	0.742656	-0.003	-0.35%
90	0.85263881	0.866139	-0.014	-1.58%

Rather than adjusting the assurance function payable at the end of year of death of the table to an assurance payable immediately on death, instead adjusting the assurance payable immediately on death of the log-normal curve to that payable at the end of year of death by the reverse of the claim acceleration formula

$$A_x \approx (1+i)^{-0.5} \bar{A}_x$$

results in the tabulated figures with similarly small absolute and relative differences;

x	payable at end of year of death			
	$A_x @ 4\%$		Absolute	Relative
	table	log-normal	difference	difference
20	0.11479347	0.107654	0.007	6.22%
30	0.16083299	0.157222	0.004	2.25%
40	0.22683616	0.227227	0.000	-0.17%
50	0.31850865	0.322502	-0.004	-1.25%
60	0.43901327	0.444319	-0.005	-1.21%
70	0.58202239	0.585546	-0.004	-0.61%
80	0.72556958	0.728086	-0.003	-0.35%
90	0.83562495	0.848832	-0.013	-1.58%

1.2.3. n-year Term $\bar{A}_{x:n|}^1$ – a benefit of 1 payable to the estate of the policyholder (x) if the policyholder does not survive n years from inception of the policy; payable at the exact moment of passing: *if the life dies within n years time, pay him 1 at that time*

$$\begin{aligned}\bar{A}_{x:n|}^1 &= \int_0^n v^t {}_t p_x \mu_{x+t} dt \\ &= \int_0^n v^t \frac{d(-{}_t p_x)}{dt} dt \\ &= -\int_0^n v^t d{}_t p_x \\ &= [-v^t {}_t p_x]_0^n - \int_0^n \delta v^t {}_t p_x dt \\ &= 1 - \delta \bar{a}_{x:n|} - {}_n E_x\end{aligned}$$

Say a life now aged 40 takes out an assurance policy covering his estate's fiduciary interests to the extent of 1 upon his passing if this occurs within 20 years from inception of the policy, with the money worth 4% per annum. The actuarial present value of this benefit, according to the log-normal mortality model, is

$$\begin{aligned}\bar{A}_{40:20|}^1 &= 1 - \delta_{4\%} \bar{a}_{40:20|} - {}_{20} E_{40} \\ &= 1 - \ln(1.04) \times 13.6179 - 0.42821 \\ &= 0.03769\end{aligned}$$

alternatively, by integration,

$$\begin{aligned}\bar{A}_{40:20|}^1 &= \frac{1}{k} \int_0^{20} 1.04^{-t} \frac{\phi([\log(72-t) - 3.40848] / 0.368547)}{0.368547(72-t)} dt \\ &= 0.03769\end{aligned}$$

where $k = \Phi([\log(72) - 3.40848] / 0.368547)$

The equivalent actuarial present value based directly on the life table is

$$\begin{aligned}A_{40:20|}^1 &= A_{40} - {}_{20} E_{40} \times A_{60} \\ &= \frac{M_{40}}{D_{40}} - \frac{D_{60}}{D_{40}} \times \frac{M_{60}}{D_{60}} \\ &= \frac{M_{40} - M_{60}}{D_{40}} \\ &= \frac{4581.74 - 3788.82}{20198.45} \\ &= 0.039256\end{aligned}$$

Accelerating the claim payment to the midpoint of the year as an approximation to a policy of immediate payment upon claim, rather than a deferred end of year claim payment, by removing one half of one years worth of discounting with the assumption that claims occur uniformly over each year of age (midyear on average),

$$\begin{aligned}\bar{A}_{40:20|}^1 &= (1.04)^{0.5} \times A_{40:20|}^1 \\ &= (1.04)^{0.5} \times 0.039256 \\ &= 0.04003\end{aligned}$$

For the premium for the 20 year term insurance for a life aged 40, $\bar{A}_{40:20|}^1$, the log-normal model has placed the value 0.03769 per unit sum insured whilst the life-table has valued it at 0.04003 per unit sum insured. The log-normal model has under-estimated the value by roughly 6%. This is a

⁶ Neill, A. (1979), *Life Contingencies*. Heineman Publishing. London.

little disappointing (but only a little, subject to further research) and has occurred because of the pure endowment, which adjusts the endowment insurance to term insurance, and the divergence between the table and log-normal curve at the earlier ages. This is rectifiable with a normal curve for the non-biological mortality at the younger ages spliced into the log-normal curve of biological mortality.

1.2.4. n -year Endowment $\bar{A}_{x:n|}$ – a benefit of 1 payable to the estate of the policyholder (x) if the policyholder does not survive n years from inception of the policy or at the end of n years if the policyholder survives the term of n years: *pay the life 1 whenever he dies, else pay him 1 on survival n years*

$$\bar{A}_{x:n|} = \bar{A}_{x:n|}^1 + {}_nE_x = 1 - \delta \bar{a}_{x:n|}$$

Say the life aged 40 from the previous example takes out a similar assurance policy with an addition savings component that, if he survives the 20 year term, then he will receive the benefit of 1. That is, he purchases a 20 year endowment policy that protects his fiduciary estate's interests against his passing if this occurs before he is 60, or else he will receive the benefit on survivorship to his 60th birthday. The actuarial present value of this endowment policy is, according to the log-normal mortality curve

$$\begin{aligned}\bar{A}_{40:20|} &= \bar{A}_{40:20|}^1 + {}_{20}E_{40} \\ &= 0.03769 + 0.42821 \\ &= 0.465896\end{aligned}$$

or alternatively,

$$\bar{A}_{40:20|} = 1 - \delta_{4\%} \bar{a}_{40:20|} = 1 - \ln(1.04) \times 13.6179 = 0.465896$$

The actuarial present value according to the life table is

$$\begin{aligned}\bar{A}_{40:20|} &= \sqrt{1.04} \frac{M_{40} - M_{60}}{D_{40}} + \frac{D_{60}}{D_{40}} \\ &= \frac{\sqrt{1.04}(4581.74 - 3788.82) + 8630.31}{20198.45} \\ &= 0.46731\end{aligned}$$

For the 20 year endowment insurance for a life aged 40, $\bar{A}_{40:20|}$, the log-normal curve results in a premium of \$0.465896 per \$1 sum insurance while the premium based on the lifetable is \$0.46731, a proportional discrepancy of only 0.3%. This, unlike the term insurance, is a very good result.

1.3. NET PREMIUMS, P

Each of the alternative pure premiums for standard fully continuous assurances can be valued with combinations of the assurance 'A' functions and annuity 'a' functions as per standard actuarial theory. Premiums paid at discrete points in time can be valued by the log-normal mortality methodology by converting the continuous life-annuities to life-annuities paid in advance by adding $\frac{1}{2}$ and to life-annuities paid in arrears by subtracting $\frac{1}{2}$ (or with the additional correction of $1/12$ of the sum of the force of interest and force of mortality). Similarly, if claims are to be paid at the end of the year that a claim is made, then claim deferment can be taken into account by reducing the amount of discounting at v for one half of one year.

Consider a life aged (x) who takes out a whole of life insurance policy, payable immediately upon claim, with premiums payable continuously while he is alive. The annual premium rate per annum is

$$\bar{P}_x(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x} = \frac{1}{\bar{a}_x} - \delta$$

Say the life is aged 40 and money is worth 4%. The annual premium rate per unit sum insured according to the log-normal theory of mortality is

$$\bar{P}_{40}(\bar{A}_{40}) = \frac{1}{\bar{a}_{40}} - \delta = \frac{1}{19.5888} - \ln(1.04) = 0.011829$$

while according to the table, with the approximations to a continuous mortality function, the annual premium rate per unit sum insured is

$$\bar{P}_{40}(\bar{A}_{40}) = \frac{1}{\bar{a}_{40}} - \delta = \frac{1}{19.59887} - \ln(1.04) = 0.011802$$

The net annual premium for the whole of life policy for a life aged 40 $\bar{P}_{40}(\bar{A}_{40})$ is \$0.01183 per annum per \$1 sum insured based on the log-normal curve while it is \$0.01180 for the lifetable; the proportional difference between the two premium calculations is near negligible at 0.2%.

continuous premiums, benefits payable at moment of death				
x	$\bar{P}(\bar{A}_x)@4\%$		Absolute difference	Relative difference
	table	log-normal		
20	0.00519998	0.00483657	0.000363409	6.99%
30	0.00769465	0.0074888	0.000205844	2.68%
40	0.01180264	0.01182895	-2.63115E-05	-0.22%
50	0.0188673	0.01921965	-0.000352351	-1.87%
60	0.03179336	0.0324958	-0.000702448	-2.21%
70	0.0572827	0.05814375	-0.00086105	-1.50%
80	0.11167555	0.1131856	-0.001510047	-1.35%
90	0.2269329	0.25377644	-0.02684354	-11.83%

The equivalent table of figures of premiums paid in advance and insurance paid at the end of the year of death presents a slightly closer correspondence between the net premium of the table derived from the Human Mortality Database (the q_x) and the net premium of the log-normal curve;

premiums in advance, benefits payable at end of year of death				
x	$P(A_x)@4\%$		Absolute difference	Relative difference
	table	log-normal		
20	0.00498769	0.00464006	0.000347629	6.97%
30	0.00737146	0.00717509	0.000196368	2.66%
40	0.01128411	0.01130927	-2.51581E-05	-0.22%
50	0.01797577	0.01830837	-0.000332602	-1.85%
60	0.03009898	0.03075361	-0.000654639	-2.17%
70	0.05355664	0.05433898	-0.000782342	-1.46%
80	0.10168888	0.10298581	-0.00129693	-1.28%
90	0.19552494	0.215968	-0.020443064	-10.46%

1.4. INSURANCE RESERVES, V

The establishment of actuarial reserves for individual policies and blocks of policies, according to the log-normal theory of mortality, is possible according to the same methods by combining the required assurance and annuity, 'A' and 'a', functions as per conventional actuarial life insurance reserving methods. Considering a reserve for an insurance coverage for a life aged x at inception. The prospective reserve for this policy at duration t is

$${}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x(\bar{A}_x)\bar{a}_{x+t} = 1 - (\delta + \bar{P}_x(\bar{A}_x))\bar{a}_{x+t}$$

Taking an example consistent with those looked at so far; considering a life that took out a whole life policy at age 40, paying continuous level premiums whenever the policy remains in force. The prospective reserve for this policy at duration 10 is

$${}_{10}\bar{V}_{40} = \bar{A}_{50} - \bar{P}_{40}(\bar{A}_{40})\bar{a}_{50} = 1 - (\delta + \bar{P}_{40}(\bar{A}_{40}))\bar{a}_{50}$$

Assuming log-normal mortality; recalling that the net level continuously paid premium for this policy is $\bar{P}_{40}(\bar{A}_{40}) = 0.011829$, the value of this reserve is

$$\begin{aligned}{}_{10}\bar{V}_{40} &= 1 - (\delta + \bar{P}_{40}(\bar{A}_{40}))\bar{a}_{50} \\ &= 1 - 17.1115(\ln(1.04) + 0.011829) \\ &= 0.1264634\end{aligned}$$

According to the table on which the log-normal mortality parameters were estimated, the value of the reserve is

$$\begin{aligned} {}_{10}\bar{V}_{40} &= 1 - (\delta + \bar{P}_{40}(\bar{A}_{40}))\bar{a}_{50} \\ &= 1 - 17.21526(\ln 1.04 + 0.011803) \\ &= 0.12162 \end{aligned}$$

The difference in the reserve values, for a whole of life policy 10 years into the term for a life aged 40 at inception ${}_{10}\bar{V}_{40}$ is proportionately about 4% different, 0.1264 per sum insured for the log-normal curve and 0.1216 for the table.

10th year reserve continuous premiums, benefits payable at moment of death

$x, x+t$	${}_t\bar{V}_x$ @ 4%		Absolute difference	Relative difference
	table	log-normal		
20, 30	0.05317378	0.05678137	-0.003607588	-6.78%
30, 40	0.08051208	0.08501837	-0.004506288	-5.60%
40, 50	0.12161984	0.12646587	-0.004846022	-3.98%
50, 60	0.18202112	0.18512008	-0.003098962	-1.70%
60, 70	0.26412892	0.26342234	0.000706584	0.27%
70, 80	0.36046521	0.36115161	-0.000686397	-0.19%
80, 90	0.43304822	0.47983655	-0.046788327	-10.80%

10th year reserve premiums in advance, benefits payable at end of year of death

$x, x+t$	${}_tV_x$ @ 4%		Absolute difference	Relative difference
	table	log-normal		
20, 30	0.05200991	0.05554811	-0.003538196	-6.80%
30, 40	0.07865321	0.08306438	-0.004411172	-5.61%
40, 50	0.11856799	0.12328919	-0.004721201	-3.98%
50, 60	0.17682488	0.17980511	-0.002980234	-1.69%
60, 70	0.25492424	0.2541512	0.00077304	0.30%
70, 80	0.34343273	0.3439219	-0.000489165	-0.14%
80, 90	0.40103197	0.44406069	-0.043028716	-10.73%

Consideration and calculations of insurance reserves are best viewed in the context of simulation for which example systems are demonstrated in the third part of this document.

Part 2: Distribution of life contingent cash flows according to the log-normal theory of mortality

This section looks at the distribution of mortality contingent cash flows; insurances, annuities, premiums, and reserves. The moments of each are initially looked at to give an indication of shape and then the distribution of the random variable of the cash flow function is derived from the technique of distribution functions. The distribution, the density and cumulative density for the cash flow function is derived for log-normal mortality from which confidence intervals are obtained. It is the purpose of this topic to talk about the distribution of possible outcomes for each of the cash flow functions; importantly, what is and what is not a conservative valuation basis for each of the cash flow functions in terms of the direction of skew, and when the expected value is and when it is not a conservative measure of center. In the context of the log-normal theory of mortality for which the future lifetime variable is log-normal, the distributions of the mortality contingent cash flows are double log-normal (except the premium ratio which is logistic log-normal); exponential exponential normal curves $\exp(\exp(N(0,1)))$

$$A, a \sim \gamma + \phi \exp(\alpha + \beta \exp(N(0,1)))$$

This section is a precursor and foundation to the next topic of simulation which will become an essential element to becoming accustomed with the random nature of the lifetime contingent functions whereby the future can be experienced thousands of times simply and easily beforehand (before

setting premiums and **designing a system and then waiting to see, after only getting ‘one chance at it’, whether luck has been on your side).**

This topic concentrates on whole of life insurance with premiums payable whenever a policy is in force, i.e., those pertaining to

$$\bar{A}_x \quad \text{and} \quad \bar{a}_x$$

It is but a small step to adjust the mathematics to term and endowment insurance with any variation of premiums payable or deferral periods, i.e., policies based on

$$\bar{A}_{x:\overline{n}|}, \bar{A}_{x:\overline{n}|}, {}_m|\bar{A}_{x:\overline{n}|}, \text{ etc,} \quad \text{and} \quad \bar{a}_{x:\overline{n}|}, {}_m|\bar{a}_{x:\overline{n}|}, \text{ etc}$$

and joint-life statuses such as

$$\bar{A}_{x,y} \quad \text{and} \quad \bar{a}_{x,y}$$

This document does not consider these functions nor the associated premiums or reserves because to do so would result in something of the order of a textbook [which one of you may like to write]. For the more complicated functions, it is only necessary to include a minimum or maximum function⁷ applied to the future lifetime variable with any necessary conditioning, for example,

$$\min(T_{40}, 10) \quad T_0 | T_0 \geq 40$$

for a 10 year term. It is the purpose, moreover, of this section to begin a detailed discussion of the statistical nature of premiums, claims, and reserves by reference to distribution theory utilizing the smooth and continuous curve of the log-normal theory of mortality. For mathematics that includes non-biological mortality at the younger ages in early adulthood, a normal should be mixed into the log-normal curve (with maximum likelihood) or the mathematics adjusted to an appropriate standard mortality table.

Because the underlying random variable of assurance functions is simpler than those of life annuities, assurance functions are considered first.

2.1. Insurance function, A , with log-normal mortality

2.1.1. Moments, A

For a traditional whole life assurance function, the first moment is the ordinary actuarial present value;

$$E(v^{T_x}) = \int_0^{\omega-x} v^t {}_tP_x \mu_{x+t} dt = \bar{A}_x$$

The second raw moment is another whole life assurance function evaluated at an annual interest rate at two years worth of the original rate for which the superscript⁸ on the left hand side of the assurance function denotes, i.e.,

$$E(v^{2T_x}) = \int_0^{\omega-x} v^{2t} {}_tP_x \mu_{x+t} dt = {}^2\bar{A}_x$$

Similarly for the k^{th} raw moment of the traditional whole life assurance function; the function is a standard assurance function evaluated at an interest rate of k -compounding years worth of the original compound rate

$$E(v^{kT_x}) = \int_0^{\omega-x} v^{kt} {}_tP_x \mu_{x+t} dt = {}^k\bar{A}_x$$

Consider a life now aged 40 who covers his estate's fiduciary interests against his passing to the amount of 1, whenever this may happen. The present value of the policy was evaluated in the first section

$$\bar{A}_{40} = 0.231715$$

and the second raw moment according to the log-normal model, by numerical integration, is

$${}^2\bar{A}_{40} = 0.070118$$

The variance is then

⁷ The Actuarial Education Company, ActEd, (2004) *Course notes for subject 104*. Faculty and Institute of Actuaries, Great Britain.

⁸ although not technically standard International Actuarial Notation, it may well become so.

$$^2\bar{A}_{40} - (\bar{A}_{40})^2 = 0.070118 - 0.231715^2 = 0.016426$$

and the standard deviation is

$$\sqrt{{}^2\bar{A}_{40} - (\bar{A}_{40})^2} = 0.128166$$

The third raw moment is

$$^3\bar{A}_{40} = 0.028638$$

and the third central moment is

$$^3\bar{A}_{40} - 3^2\bar{A}_{40}\bar{A}_{40} + 2(\bar{A}_{40})^3 = 0.004778$$

The cube root of this moment is

$$\sqrt[3]{^3\bar{A}_{40} - 3^2\bar{A}_{40}\bar{A}_{40} + 2(\bar{A}_{40})^3} = 0.168425$$

and scaled by the standard deviation,

$$\sqrt[3]{^3\bar{A}_{40} - 3^2\bar{A}_{40}\bar{A}_{40} + 2(\bar{A}_{40})^3} / \sqrt{{}^2\bar{A}_{40} - (\bar{A}_{40})^2} = 1.30166$$

Clearly, a skewed distribution; the further raw moments to order 6 are

$$^4\bar{A}_{40} = 0.015222$$

$$^5\bar{A}_{40} = 0.009755$$

$$^6\bar{A}_{40} = 0.00704$$

The other moment indices of distributional shape, notably the central moments and the k^{th} roots of the scaled k^{th} central moments, are tabulated and can alternatively be calculated numerically by the formula (which was how the numbers in the table were calculated)

$$E\left((v^{T_x} - \bar{A}_x)^k\right) = \int_0^{\omega-x} (v^{T_x} - \bar{A}_x)^k {}_tP_x \mu_{x+t} dt$$

Central moments and k^{th} roots of normalized central moments

k	$E[(v^{T_{40}} - \bar{A}_{40})^k]$	$\sqrt[k]{E[(v^{T_{40}} - \bar{A}_{40})^k]} / \sigma$
1	0	0
2	0.016426	1
3	0.004778	1.31412
4	0.002619	1.76515
5	0.001443	2.10907
6	0.00087	2.41090

2.1.2. Distribution, A

The exact distribution of the insurance variable v^{T_x} is now derived based on the statistical transformation of the lifetime random variable and the log-normal curve of the future lifetime variable for the log-normal theory. Consider the random variable T_x and the related random variable of the assurance providing lifetime cover to a life now aged x , v^{T_x} , which is an exponential transformation of T_x and is hence one-to-one. Letting A denote the random variable v^{T_x} , the random variable that denotes the present value of the insurance claim under a whole of life policy

$$A = v^{T_x}$$

The domain of the assurance random variable is then

$$0 < T_x < \infty$$

$$v^\infty < v^{T_x} < v^0$$

$$0 < v^{T_x} < 1$$

Theoretically, the maximum loss present value to the issuer is 1 if there is a claim immediately and the minimum possible loss is 0 if the policyholder lives forever. Assuming the terminal limit to lifespan of ω the limits become

$$0 < T_x < \omega - x$$

$$v^{\omega-x} < v^{T_x} < 1$$

That is, a minimum possible loss of $v^{\omega-x}$ because of the upper limit to lifespan. The distribution function of A , the probability that the present value of the insurance payout will be less than the specified value y , is then

$$\begin{aligned} F_A(y) &= \Pr(A \leq y) \\ &= \Pr(v^{T_x} \leq y) \\ &= \Pr(T_x \log v \leq \log y) \\ &= \Pr(-T_x \delta \leq \log y) \\ &= \Pr(T_x \geq -(1/\delta) \log y) \\ &= S_{T_x}(-(1/\delta) \log y) \\ &= \frac{S_{T_0}(x - (1/\delta) \log y)}{S_{T_0}(x)} \end{aligned}$$

In terms of traditional actuarial functions, the distribution function of the assurance can be expressed as

$$F_A(y) = {}_wP_x = {}_{x+w}P_0 / {}_xP_0 \quad \text{where } w = -(1/\delta) \log y$$

which can be calculated from any of the conventional actuarial methods, with an assumption for non-integral values of age (linear, exponential, or harmonic interpolation of the l_x); such an assumption is not required for a log-normal mortality curve as considered here as the log-normal theory has a continuous density. Differentiating the distribution function for the density function of the assurance, A , the probability that the present value of the insurance payout will equal y

$$\begin{aligned} f_A(y) &= dF_A(y) / dy \\ &= dS_{T_x}(-(1/\delta) \log y) / dy \\ &= dS_{T_x}(u) / du \times d(-(1/\delta) \log y) / dy \\ &= \frac{1}{\delta y} f_{T_x}(-(1/\delta) \log y) \\ &= \frac{1}{\delta y} \frac{f_{T_0}(x - (1/\delta) \log y)}{S_{T_0}(x)} \end{aligned}$$

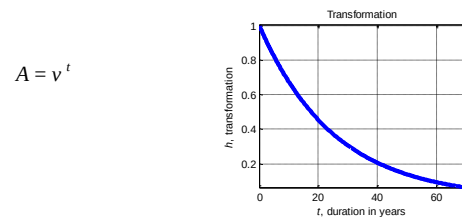
In terms of traditional actuarial functions, the density of the assurance can be expressed as

$$f_A(y) = \frac{1}{\delta y} {}_wP_x \mu_{x+w} \quad \text{where } w = -(1/\delta) \log y$$

and where $1/\delta y$ is the Jacobian of the transform from T to A . Given that T is determined by the log-normal theory of mortality, the distribution function and density functions of the traditional whole life policy issued to a life aged x is then

$$\begin{aligned} F_A(y) &= \frac{\Phi([\log(\omega - x + (1/\delta) \log y) - \mu]/\sigma)}{\Phi([\log(\omega - x) - \mu]/\sigma)} \quad y \in (v^{\omega-x}, 1) \\ f_A(y) &= \frac{1}{\delta y} \frac{\phi([\log(\omega - x + (1/\delta) \log y) - \mu]/\sigma)}{\sigma(\log(\omega - x + (1/\delta) \log y) \Phi([\log(\omega - x) - \mu]/\sigma))} \quad y \in (v^{\omega-x}, 1) \end{aligned}$$

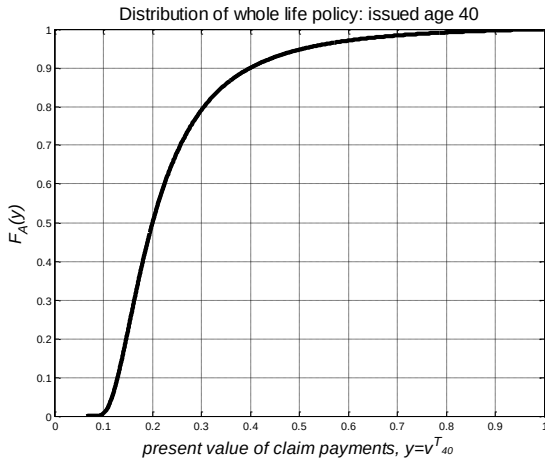
Which are both everywhere continuous and relatively easy to program. The transformation of the lifetime variable to the cash flow variable for the present value of insurance A for a life aged 40 with money worth 4% per annum is



$$A = v^t$$

The distribution function of the insurance A with log-normal mortality is then

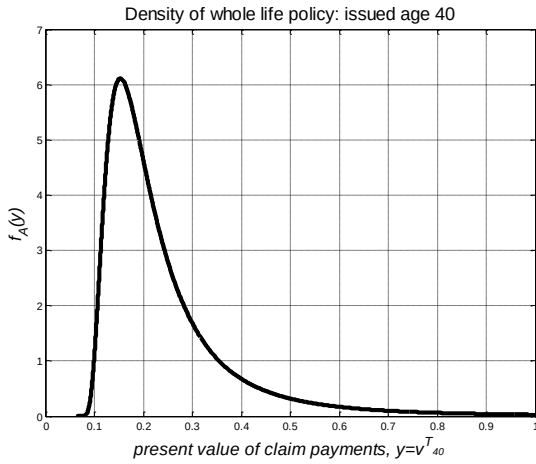
$$F_A(y) = \frac{\Phi([\log(72 + \delta_4^{-1} \log y) - 3.4085]/0.36855)}{\Phi([\log(72) - 3.4085]/0.36855)} \quad y \in (0.0594, 1)$$



and the probability density function of A with log-normal mortality is

$$f_A(y) = \frac{1}{\delta_{4\%} y} \frac{\phi\left(\frac{[\log(72 + \delta_{4\%}^{-1} \log y) - 3.4085]}{0.36855}\right)}{\sigma(72 + 1.04^{-1} \log y) \Phi\left(\frac{[\log(72) - 3.4085]}{0.36855}\right)}$$

$$y \in (0.0594, 1)$$



The probability mass function is enlightening with regard to what sort of time-valued loss the policy represents to the issuer. A full density analysis can be undertaken in a similar manner as the lifetime distribution itself, for instance, modal analysis and calculation of various statistics and confidence intervals as well as running simulations. The right tail is a result of very early claims.

2.1.3. Bounds, A

Deriving confidence intervals for the present value of the insurance loss,

$$0.95 = \Pr(a < A < b)$$

or to make the interval symmetric with respect to tail probability,

$$0.025 = \Pr(A < a) = F_A(a)$$

$$0.975 = \Pr(A < b) = F_A(b)$$

Solving these probability equations results in the bounds

$$LB(A, 95\%) = \exp\left(-\delta\left[\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.025 S_{T_0}(x)\right)\right)\right]\right)$$

$$UB(A, 95\%) = \exp\left(-\delta\left[\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.975 S_{T_0}(x)\right)\right)\right]\right)$$

So the confidence interval is then (a, b) where $a = LB(A; 95\%)$ and $b = UB(A; 95\%)$,

$$\left(\frac{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.025 S_{T_0}(x)\right)\right)}{v}, \frac{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.975 S_{T_0}(x)\right)\right)}{v} \right)$$

noting that because the distribution is skewed and considering that it is large losses that are of the main concern (considering that only one tail is of importance), it may be reasonable to calculate one sided bounds. For the whole life policy with an age at issue of 40 with money worth 4% flat, the confidence interval by the log-normal lifetime function with modal matched parameters (without rounding in the mortality parameters) is calculated as

$$S_{T_0}(40) = \Phi\left(\frac{[\log 72 - 3.4085]}{0.36855}\right) = 0.99076$$

$$UB(A, 95\%) = v^{-72 + \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.975 S_{T_0}(40)\right)\right)} = 0.605442$$

$$LB(A, 95\%) = v^{-72 + \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.025 S_{T_0}(40)\right)\right)} = 0.105487$$

The confidence interval is (\$0.105487, \$0.605442) per unit sum insured. This is a concentrated distribution (because it is double log-normal), and is significantly more concentrated than, say, a normal which has 95% of its probability bulk in around 4 standard deviations. Notice that, since the present value of the insurance loss is skewed upwards, the expected value of \$0.2317 and the standard deviation of the benefit of \$0.1282 does not give a complete picture of the present value of loss that the policy represents to the insurance company. The lower bound is less than one standard deviation below the net premium (very close to the net premium) while the upper bound is almost three standard deviations above⁹ because of the fact that the A function is right skew

$$Mean(A) > Median(A) > Mode(A)$$

which is reverse in direction from the future lifetime variable

$$Mean(T_0) < Median(T_0) < Mode(T_0)$$

and remains so with insured lives for ages before the mode, $mode(T_0)$. Just before the mode of T_0 (about 85), the skew of the conditional distribution of T is reversed as is that of A .

Given that the log-normal theory of mortality provides an accurate description of the mortality variable,

- There is a 2.5% probability that the present value of the claim benefit will be less than \$0.1055 per unit sum insured (for long lived policyholders);
- There is a 2.5% probability that the present value of the claim benefit will exceed \$0.6054 per unit sum insured (for short lived policyholders).

These intervals are important to the function of reserving for future claims and for establishing the extent of a margin to which to add to a net premium to cover for variations in the present value of the claim. Such intervals are also of benefit to the analysis of profit and ex-post examination of actual outcomes verse the expected outcomes at the time the premiums were calculated. This is given further consideration in the next part which concerns simulating lifetimes and mortality contingent cash flows. The recurring lesson is that premiums must be saved to pay claims and any company not willing to do so should not be writing long-term insurance.

The formulae for the percentile bands provide the structure for the expression for the distributions median (since each are based on percentiles), evaluated for a probability of 50%. The formula for the median is

$$\begin{aligned} Median(A) &= \exp\left(-\delta\left[\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.50 S_{T_0}(x)\right)\right)\right]\right) \\ &= v^{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.50 S_{T_0}(x)\right)\right)} \end{aligned}$$

since

$$Median(T_x) = \omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.50 S_{T_0}(x)\right)\right)$$

is the quartile of the log-normal future lifetime distribution T_x . For the example of a life aged 40 with money worth 4% per annum, the median is

$$\begin{aligned} Median(A) &= v_{4\%}^{112 - 40 - \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.025 S_{T_0}(40)\right)\right)} \\ &= 0.19326 \end{aligned}$$

⁹ The result based on the table that the log-normal model is approximating is very similar.

or \$0.1933 per dollar insured. This is less than the expected value, or net single premium, of \$0.2317 per dollar insured. This reflects the conservative basis of the traditional actuarial method for insurance, as it is more likely than not, that any particular insurance policy of this type will cost the insurer less than the single net premium. The median value of \$0.1933 reflects the cost if the insured lives to the median age of the mortality variable conditional on survival to age 40, T_{40} ; the 50% likely premium, i.e., for an insured whom 50% of the other insureds die before him and 50% after.

2.2. Annuity function, a , with log-normal mortality

2.2.1. Moments, a

A life annuity is a geometric progression of the life insurance function and whence summed, simply a change of scale and location of the insurance function; therefore, the moments of the life annuity are obtained directly from the corresponding life assurance function by reference to the linear relationship between these two random variables linked through the mortality variable. The random variable for the life annuity is

$$\bar{a}_{T_x|} = \frac{1-v^{T_x}}{\delta}$$

So the first raw moment of the present value of the payment stream is

$$E(\bar{a}_{T_x|}) = E\left(\frac{1-v^{T_x}}{\delta}\right) = \frac{1-\bar{A}_x}{\delta}$$

Similarly, the second raw moment of the life annuity is

$$E(\bar{a}_{T_x|}^2) = E\left(\left(\frac{1-v^{T_x}}{\delta}\right)^2\right) = E\left(\frac{1-2v^{T_x}+v^{2T_x}}{\delta^2}\right) = \frac{1-2\bar{A}_x+{}^2\bar{A}_x}{\delta^2}$$

and the third,

$$E(\bar{a}_{T_x|}^3) = E\left(\left(\frac{1-v^{T_x}}{\delta}\right)^3\right) = \frac{1-3\bar{A}_x+3{}^2\bar{A}_x-{}^3\bar{A}_x}{\delta^3}$$

where ${}^2\bar{A}_x$ and ${}^3\bar{A}_x$ are the insurance functions evaluated at annual effective interest rates equal to 2 and 3 compounding years worth of the original rate respectively. The general k^{th} raw moment is derived from the binomial expansion of the annuity function

$$\begin{aligned} E(\bar{a}_{T_x|}^k) &= E\left(\left(\frac{1-v^{T_x}}{\delta}\right)^k\right) \\ &= E\left(\frac{1}{\delta^k} \sum_{r=0}^k (-1)^r {}^kC_r v^{rT_x}\right) \\ &= \frac{1}{\delta^k} \sum_{r=0}^k (-1)^r {}^kC_r {}^r\bar{A}_x \end{aligned}$$

${}^r\bar{A}_x$ is the insurance function evaluated at an annual effective interest rate equal to r years compounding worth of the original rate. From the earlier raw moments of the assurance function for the whole life policy for a life aged 40 at inception with money worth 4% flat and mortality according to the log-normal theory of mortality, the mean of the corresponding annuity is, as before,

$$E(\bar{a}_{T_{40}|}) = \bar{a}_{40} = \frac{1-\bar{A}_{40}}{\delta} = \frac{1-0.231715}{\delta} = 19.5888$$

or a cost of \$19.5888 per dollar benefit per year of living. The median according to log-normal mortality is also a linear function of the median of the insurance

$$\text{Median}(\bar{a}_{T_x|}) = \frac{1-\text{Median}(\bar{A}_{T_x|})}{\delta}$$

From the analytical expression derived in the following section for confidence intervals for the present value of the annuity stream

$$\text{Median}(\bar{a}_{T_x|}) = \frac{1-v^{\omega-x-\exp(\mu+\sigma\Phi^{-1}(0.025S_{T_0}(x)))}}{\delta}$$

For the annuity being considered for a life aged 40, the median value is

$$\begin{aligned} \text{Median}(\bar{a}_{T_{40}|}) &= \frac{1-v^{112-40-\exp(3.4085+0.36855\Phi^{-1}(0.50S_{T_0}(x)))}}{\delta_{4\%}} \\ &= 20.5692 \end{aligned}$$

or, \$20.5692 per dollar benefit per year of living. This number is larger than the weighted average so it should be expected that the skew of the present value of the pension cost is negative because the annuity transform of the mortality variable is a linear location-scale transform of the insurance transform of the mortality variable, with reflection built into the scale coefficient, namely, $-1/\delta$. Considering second order distributional features, the second raw moment is

$$E(\bar{a}_{T_{40}|}^2) = \frac{1-2\bar{A}_{40}+{}^2\bar{A}_{40}}{\delta^2} = \frac{1-2(0.231715)-0.070118}{\delta^2} = 394.398$$

The second central moment, the variance, of the annuity stream is

$$\text{Var}(\bar{a}_{T_{40}|}) = E(\bar{a}_{T_{40}|}^2) - E(\bar{a}_{T_{40}|})^2 = 394.398 - 19.5888^2 = 10.6786$$

and the standard deviation is

$$\text{SD}(\bar{a}_{T_{40}|}) = \sqrt{\text{Var}(\bar{a}_{T_{40}|})} = \sqrt{10.6786} = 3.2678$$

Considering third order distributional features, the third raw moment is

$$\begin{aligned} E(\bar{a}_{T_{40}|}^3) &= \frac{1-3\bar{A}_{40}+3({}^2\bar{A}_{40})-{}^3\bar{A}_{40}}{\delta^3} \\ &= \frac{1-3(0.231715)+3(0.070118)-0.028638}{\delta^3} \\ &= 8064.94 \end{aligned}$$

and skewness

$$\begin{aligned} \text{Skew}(\bar{a}_{T_{40}|}^3) &= E\left(\left(\bar{a}_{T_{40}|} - \bar{a}_{40}\right)^3\right) \\ &= E(\bar{a}_{T_{40}|}^3) - 3E(\bar{a}_{T_{40}|}^2)\bar{a}_{40} + 2(\bar{a}_{40})^3 \\ &= 8064.94 - 3(394.398) + 2(19.5888)^3 \\ &= -79.19 \end{aligned}$$

which is negatively (left) skewed. The tail of the distribution runs in the same direction as the mortality variable, the opposite to that of the insurance function. Note that unsigned skew times the third power of the force of interest, $79.19 \times \delta^3 = 0.004778$, is the skew of the associated insurance function. The signed cube root of the skewness is

$$-\sqrt[3]{\text{Skew}(\bar{a}_{T_{40}|}^3)} = -\$4.29$$

and de-scaled so that the shape measure is independent of the spread of the distribution, the third order moment shape index is

$$\frac{-\sqrt[3]{\text{Skew}(\bar{a}_{T_{40}|}^3)}}{\sqrt{\text{Var}(\bar{a}_{T_{40}|}^3)}} = \frac{-\$4.2943}{\$3.2678} = -1.31412$$

which is the same as that for the insurance distribution with a change of sign. All of the normalised indices of distributional shape, are exactly the same as those for the insurance function; apart from positioning and spread (and reflection), the distributions are identical. The change in the sign of skew, that of reflection, means that conservatism for outflow [say A] becomes conservatism for inflow [say a].

The central moments can be calculated numerically by the formula

$$E\left(\left[(1-v^{T_x})/\delta - \bar{a}_x\right]^k\right) = \int_0^{\omega-x} \left((1-v^{T_x})/\delta - \bar{a}_x\right)^k {}_tP_x \mu_{x+t} dt$$

which are exactly those of the insurance function with a scale to the k^{th} power of the force of interest

$$E\left(\left[(1-v^{T_x})/\delta - \bar{a}_x\right]^k\right) = \frac{(-1)^k}{\delta^k} \int_0^{\omega-x} (v^{T_x} - \bar{A}_x)^k {}_tP_x \mu_{x+t} dt$$

so that the roots of the central moments normalized by the standard deviation are exactly the same as those of the insurance function; the same relative skew, kurtosis, etc with odd moments having a change of sign.

Central moments and k th roots of normalized central moments

k	$E[(\bar{a}_{T_{40}} - \bar{a}_{40})^k]$	$\text{sign}(\zeta_k) \sqrt[k]{ \zeta_k }$
1	0	0
2	10.67858	1
3	-79.19011	-1.31412
4	1107.00653	1.76515
5	-15550.37690	-2.10907
6	239121.22672	2.41090

Considering fourth order distributional features, the fourth central moment (kurtosis) is 1107.01. Normalising by the distributions spread, in this case by the square of the variance, the coefficient of kurtosis is 9.71 (and a cube root of 1.765); this heavy kurtosis is because the distribution is a double log-normal. This is over three times that of a standard normal. The distribution is heavily leptokurtic and hence has a sharp and concentrated peak meaning that the probability has a heavy concentration over a local range of the domain. Realisations of this random variable will frequently take values near the mode. Discrepant values, differing from the majority will occur much less frequently. When discrepant realisations do occur, they will generally be smaller due to the negative skew. In practical terms, an insurer or some other institution providing life-time annuities need to be most concerned about where this mode is, since wherever it is, practically it alone will determine the cost of almost all of the policies, and hence the ultimate cost of the aggregate portfolio. Larger spread than factored into the original valuation will tend to pull the cost per annuity downwards. So controlling for location, larger variation in the mortality variable is good for writers of annuities and smaller more stable variation is good for, say, insurers accepting level premiums as ongoing annuity payments. There are some ethical considerations here as adverse selection of a company against an unsophisticated policyholder (rather than the reverse for insurance) is a very bad thing and definitely needs to be borne in mind by members of an ethical and responsible profession when there is a conflict of interest between the profit motive of limited liability and the wellbeing and longevity of annuity holders and pensioners. This does not arise in the context of insurance where there is a vested interest in survivorship of policyholders.

2.2.2. Distribution, a

Proceeding in the same manner as for the assurance policy by considering the future lifetime random variable T_x . Letting a denote the transformation of the present value of a continuous payment stream determined by the future lifetime variable

$$a = \int_0^{T_x} v^t dt = \frac{1 - v^{T_x}}{\delta}$$

The domain of the transformation of T_x becomes

$$\begin{aligned} 0 < T_x < \infty \\ v^\infty < v^{T_x} < v^0 \\ 0 < 1 - v^{T_x} < 1 \\ 0 < a < 1/\delta \end{aligned}$$

That is, the minimum value of the annuity is 0 if death occurs immediately and the maximum value of the annuity is the present value of a perpetuity $1/\delta$ if the life lives forever. Assuming all lifetimes are below the terminal age of the mortality table, or the limit to lifespan in the log-normal theory, then the upper bound is the present value of $\omega - x$ years of continuous payments

$$\frac{1}{\delta}(1 - v^{\omega-x}) = \bar{a}_{\omega-x|}$$

rather than the perpetuity so that

$$\begin{aligned} 0 < T_x < \omega - x \\ v^{\omega-x} < v^{T_x} < v^0 \\ 0 < 1 - v^{T_x} < 1 - v^{\omega-x} \\ 0 < \frac{1}{\delta}(1 - v^{T_x}) < \frac{1}{\delta}(1 - v^{\omega-x}) \\ 0 < a < \bar{a}_{\omega-x|} \end{aligned}$$

This transformation is linear exponential and, like the assurance transformation, is one-to-one. The distribution function of the life-annuity is

$$\begin{aligned} F_a(y) &= \Pr(a \leq y) \\ &= \Pr((1 - v^{T_x})/\delta \leq y) \\ &= \Pr(1 - \delta y \leq v^{T_x}) \\ &= \Pr(\log(1 - \delta y) \leq -\delta T_x) \\ &= \Pr(T_x \leq -(1/\delta)\log(1 - \delta y)) \\ &= F_{T_x}(-(1/\delta)\log(1 - \delta y)) \\ &= \frac{F_{T_0}(x - (1/\delta)\log(1 - \delta y))}{S_{T_0}(x)} \end{aligned}$$

In terms of standard actuarial notation, the distribution function of the present value of the stream of payments can be expressed as

$$\begin{aligned} F_a(y) &= {}_wq_x \\ &= 1 - {}_{x+w}p_0 / {}_x p_0 \end{aligned} \quad \text{where } w = -(1/\delta)\log(1 - \delta y)$$

which can be calculated by reference to a lifetable with an assumption for interpolating non-integral values of age. Differentiating the distribution function with respect to the argument for the density function of the present value of payments,

$$\begin{aligned} f_a(y) &= dF_a(y)/dy \\ &= dF_{T_x}(-(1/\delta)\log(1 - \delta y))/dy \\ &= dF_{T_x}(u)/du \times (-(1/\delta)\log(1 - \delta y))/dy \\ &= \frac{1}{1 - \delta y} f_{T_x}(-(1/\delta)\log(1 - \delta y)) \\ &= \frac{1}{1 - \delta y} \frac{f_{T_0}(x - (1/\delta)\log(1 - \delta y))}{S_{T_0}(x)} \end{aligned}$$

In term of standard actuarial notation, the density of the present value of the annuity stream is

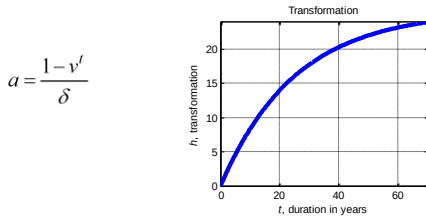
$$f_a(y) = \frac{{}_wq_x \mu_{x+w}}{1 - \delta y} \quad \text{where } w = -(1/\delta)\log(1 - \delta y)$$

where $1/(1 - \delta y)$ is the Jacobian of the transformation. For the log-normal theory of mortality and the associated log-normal curve, the distribution and density functions are

$$\begin{aligned} F_a(y) &= \frac{\Phi([\mu - \log(\omega - x + (1/\delta)\log(1 - \delta y))]/\sigma)}{\Phi([\log(\omega - x) - \mu]/\sigma)} \\ f_a(y) &= \frac{1}{1 - \delta y} \frac{\Phi([\mu - \log(\omega - x + (1/\delta)\log(1 - \delta y))]/\sigma)}{\sigma(\omega - x + (1/\delta)\log(1 - \delta y))\Phi([\log(\omega - x) - \mu]/\sigma)} \end{aligned}$$

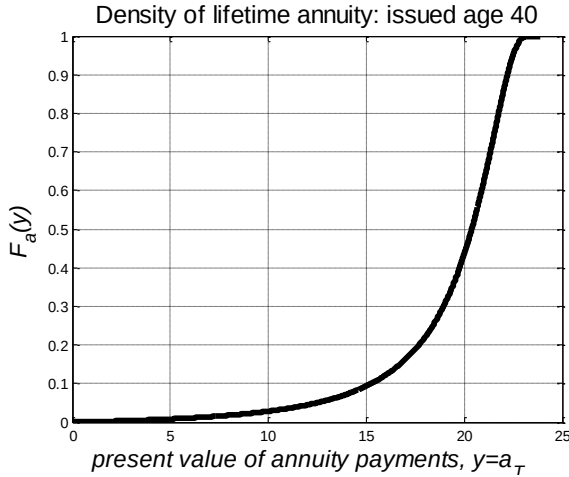
both defined for $y \in (0, \bar{a}_{\omega-x|})$.

These functions are, similar to A , everywhere continuous and easily programmable although somewhat unwieldy to manipulate by hand. For a lifetime annuity with continuous payments to a life aged 40 at inception and money worth 4% flat, the transformation of the lifetime variable to the variable of the annuity stream a is



The distribution function of the annuity a with log-normal mortality is

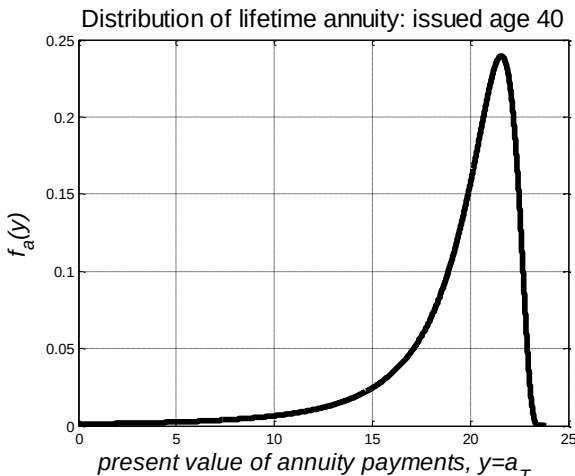
$$F_a(y) = \frac{\Phi\left(\frac{[3.4085 - \log(72 + \delta_{4\%}^{-1} \log(1 - y \delta_{4\%}))]}{0.36855}\right)}{\Phi\left(\frac{[\log(72) - 3.4085]}{0.36855}\right)} \quad y \in (0, 23.98)$$



The probability density function of the annuity a with log-normal mortality is

$$f_a(y) = \frac{1}{1 - \delta_{4\%} y} \times \frac{\Phi\left(\frac{[3.4085 - \log(72 + \delta_{4\%}^{-1} \log(1 - y \delta_{4\%}))]}{0.36855}\right)}{\sigma\left(72 + \delta_{4\%}^{-1} \log(1 - \delta_{4\%} y)\right) \Phi\left(\frac{[\log(72) - 3.4085]}{0.36855}\right)}$$

for $y \in (0, 23.98)$



This distribution is left-skewed. This distribution should be analysed carefully, in conjunction with simulation, by anyone receiving (premiums) or providing (pensions) life contingent annuity. The pure premium of the life annuity is 19.5888 which is slightly below the modal frequency.

2.2.3. Bounds, a

Deriving confidence intervals for the present value of the annuity stream with mortality contingent duration,

$$0.95 = \Pr(lb < a < ub)$$

and again making the interval symmetric with respect to tail probability,

$$0.025 = \Pr(a < lb) = F_a(lb)$$

$$0.975 = \Pr(a < ub) = F_a(ub)$$

Solving these equations results in the bounds for a with log-normal mortality

$$UB(a, 95\%) = \frac{1 - \exp\left(-\delta \left[\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.025 S_{T_0}(x)\right)\right)\right]\right)}{\delta}$$

$$LB(a, 95\%) = \frac{1 - \exp\left(-\delta \left[\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.975 S_{T_0}(x)\right)\right)\right]\right)}{\delta}$$

For the actuarial present value of a stream of payments payable continuously to a life aged 40 now, while he is alive, with money worth 4% per annum effective, the associated 95% confidence bounds are

$$S_{T_0}(40) = \Phi\left(\frac{[\log 72 - 3.4085]}{0.36855}\right) = 0.99076$$

$$UB(a, 95\%) = \frac{1 - v^{-72 + \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.025 S_{T_0}(40)\right)\right)}}{\log 1.04} = 22.8072$$

$$LB(a, 95\%) = \frac{1 - v^{-72 + \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.975 S_{T_0}(40)\right)\right)}}{\log 1.04} = 10.0599$$

which can also be derived from either the confidence intervals on the insurance function A or directly from the percentiles of the future lifetime variable T . The confidence interval for A was [0.10549, 0.60544]; the corresponding intervals for a , from those of A , are

$$[(1 - 0.60544) / \log 1.04, (1 - 0.10549) / \log 1.04] = [10.0599, 22.8072]$$

as just calculated by the formulae. There is 95% certainty that for each dollar of annual continuous payment for a lifetime past 40, the total cost to the annuity writer in terms of present value will lie on the interval (\$10.0599, \$22.8072). The width of the interval is \$12.75, which in terms of the standard deviation of \$3.27, is just under four (3.9) standard deviations wide. The probability of this distribution, like the distribution of the assurance policy, is much more concentrated than say, a normal distribution. The distribution of a is identical to that of A within a change of scale and location, with reflection. It is therefore also double log-normal like that of A . The upper bound to this interval comes very close to the maximum possible cost (of \$23.98) and is less than one standard deviation above the mean of \$19.59. The lower bound, on the other hand, is almost three standard deviations away from (below) the mean.

Given that the log-normal theory of mortality provides an accurate description of the mortality variable,

- There is a 2.5% probability that the present value of the annuity stream will exceed than \$22.81 per \$1 per annum, and;
- There is a 2.5% probability that the present value of the annuity stream will be less than \$10.06 per \$1 per annum.

Because of the direction of the skew, if a company was to sell a lot of these life annuities, each to lives aged 40, it could expect that most of the payment streams would cost around $(1 - v^{-45.5}) / \delta = \21.22 but a small number of the payment streams would cost substantially less, resulting in an average cost of around \$19.59. It is more likely than not that the present value of any payment stream for a life aged 40 will be greater than

$$\bar{a}_{40}$$

Because the a function is left skew [for ages before the mode]

$$Mean(a) < Median(a) < Mode(a)$$

since

$$\text{Mean}(A) > \text{Median}(A) > \text{Mode}(A)$$

$$\text{Mean}(A) - 1 > \text{Median}(A) - 1 > \text{Mode}(A) - 1$$

$$1 - \text{Mean}(A) < 1 - \text{Median}(A) < 1 - \text{Mode}(A)$$

$$\frac{1 - \text{Mean}(A)}{\delta} < \frac{1 - \text{Median}(A)}{\delta} < \frac{1 - \text{Mode}(A)}{\delta}$$

$$\text{Mean}(a) < \text{Median}(a) < \text{Mode}(a)$$

which is similarly the direction of the future lifetime variable

$$\text{Mean}(T_0) < \text{Median}(T_0) < \text{Mode}(T_0)$$

The skew is reversed for distributions conditional on ages past the mode of T_0 . The median of the future lifetime variable for a life aged 40 is 41.91 years so the median of a is $(1 - v^{41.91}) / \log 1.04 = 20.57$. The actuarial method is hence not conservative in the valuation of mortality contingent annuities given that the median value is around a dollar higher than the weighted mean, a function of \$19.59. There is consistency in conservatism only in the context of the actuarial task to value premiums for insurance for which the annuity pertains to inflow of (unearned) premium revenue. The method of valuing the premium [income] stream by weighted mean value leads to a reciprocal that is conservative. Together with a conservative valuation on the insurance component, the pricing of the annual premium is therefore consistent with a conservative valuation principle. It is noted that pricing by the mean is NOT a conservative basis with respect to outflow, i.e., for setting a premium to make payments for the provision of annuities (because of the direction of the skew), since the mean is the smallest measure of center. This is why longevity risk can pose such a significant risk from entities that write annuities who price with a functions, since it more likely than not that the premium charged will be insufficient to cover the benefits provided [net of the profit margin]. This is an overwhelming reason for simulating the cashflows with respect to [simulated] actual-vs-expected.

2.3. Net Premium, P , with log-normal mortality

2.3.1. Distribution, P

Analysing premiums stochastically allows entities that write insurance business to get a better understanding of the adequacy of the premium rates that they may charge on a particular type of policy, and how the net premium based on the actuarial present value compares to the possible fluctuation in the ratio of financial outflow to financial inflow - the ratio on which premiums are based. A premium is the ratio of the insurance benefits to the stream of premiums paid, by reference to the average values, for the insurance and hence analysing the ratio stochastically may be helpful in deciding on the size of a margin added to the net premium in terms of bounds of probability. For a whole life insurance, the distribution of the insurance is right skew, the distribution of the annuity is left skew so that the reciprocal of the distribution of the annuity is right skew, and the ratio of the two A to a (a logit of the T period discount factor) is whence right skew with a more concentrated mode.

The transformation method of distribution functions, as with the preceding sections for A and a , is used to derive of the probability distribution for the net premium as a random function of the ratio of the present value of the claim payments per unit sum insured to the present value of premium income of 1 per annum. Letting P denote the transformation of the future lifetime, or mortality, random variable

$$P = \frac{1}{(1 - v^{T_x})/\delta} - \delta = \frac{v^{T_x}}{(1 - v^{T_x})/\delta} = \frac{\delta v^{T_x}}{1 - v^{T_x}}$$

Hence, the premium is a logistic function of the future lifetime variable, a logistic transform of a log-normal curve in the context of the log-normal theory of mortality being considered here. The domain of the logistic function of T_x , the outflow-inflow ratio, is

$$0 < T_x < \infty$$

$$0 < v^{T_x} < 1$$

$$0 < 1 - v^{T_x} < 1$$

$$1 < 1/(1 - v^{T_x}) < \infty$$

$$\delta < \delta/(1 - v^{T_x}) < \infty$$

$$0 < \delta/(1 - v^{T_x}) - \delta < \infty$$

$$0 < P < \infty$$

which is intuitive; the ratio is

- zero (0) if the outflow is zero or the inflow is infinite; and the ratio is
- infinite (∞) if the outflow is infinite or the inflow is zero.

In terms of an upper limit on lifespan,

$$0 < T_x < \omega - x$$

$$0 < v^{T_x} < v^{\omega-x}$$

$$\frac{\delta v^{\omega-x}}{1 - v^{\omega-x}} < P < \infty$$

The probability that the outflow-inflow ratio P will be less than p , the distribution function of the ratio, is

$$\begin{aligned} F_P(p) &= \Pr(P \leq p) \\ &= \Pr(\delta v^{T_x} / (1 - v^{T_x}) \leq p) \\ &= \Pr(1/v^{T_x} - 1 \geq \delta/p) \\ &= \Pr(1/v^{T_x} \geq 1 + \delta/p) \\ &= \Pr(-T_x \log v \geq \log(1 + \delta/p)) \\ &= \Pr(T_x \geq (1/\delta) \log(1 + \delta/p)) \\ &= S_{T_x}((1/\delta) \log(1 + \delta/p)) \\ &= \frac{S_{T_0}(x + (1/\delta) \log(1 + \delta/p))}{S_{T_0}(x)} \end{aligned}$$

In standard actuarial notation, the distribution function can be written as

$$F_P(p) = {}_wP_x = {}_{x+w}P_0 / {}_xP_0 \quad \text{where } w = (1/\delta) \log(1 + \delta/p)$$

The probability density function of the outflow-to-inflow ratio is the derivative of the distribution function with respect to p

$$\begin{aligned} f_P(p) &= dF_P(p)/dp \\ &= -dS_{T_x}((1/\delta) \log(1 + \delta/p))/dp \\ &= -dS_{T_x}(u)/du \times d((1/\delta) \log(1 + \delta/p))/dp \\ &= \frac{1}{p(p + \delta)} f_{T_x}((1/\delta) \log(1 + \delta/p)) \\ &= \frac{1}{p(p + \delta)} \frac{f_{T_0}(x + (1/\delta) \log(1 + \delta/p))}{S_{T_0}(x)} \end{aligned}$$

In actuarial notation,

$$f_P(p) = \frac{{}_wP_x {}_{x+w}H_{x+w}}{p(p + \delta)} = \frac{{}_{x+w}P_0 {}_{x+w}H_{x+w} / {}_xP_0}{p(p + \delta)} \quad \text{where } w = (1/\delta) \log(1 + \delta/p)$$

The functional forms of the distribution function and the probability density function for the log-normal theory of mortality are

$$F_P(p) = \frac{\Phi([\log(\omega - x - (1/\delta) \log(1 + \delta/p)) - \mu]/\sigma)}{\Phi([\log(\omega - x) - \mu]/\sigma)} \quad p \in \left(\frac{\delta v^{\omega-x}}{1 - v^{\omega-x}}, \infty \right)$$

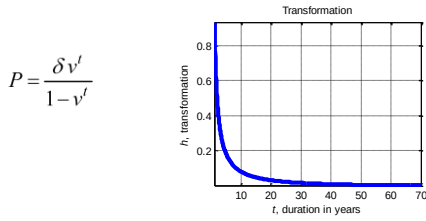
and

$$\begin{aligned} f_P(p) &= \frac{1}{p(p + \delta)} \frac{\phi([\log(\omega - x - (1/\delta) \log(1 + \delta/p)) - \mu]/\sigma)}{\sigma(\omega - x - (1/\delta) \log(1 + \delta/p)) \Phi([\log(\omega - x) - \mu]/\sigma)} \\ \text{for } p &\in \left(\frac{\delta v^{\omega-x}}{1 - v^{\omega-x}}, \infty \right) \end{aligned}$$

Considering the whole life policy for a life now aged 40 for the cost of the claim net of continuous premium payable while the policy is in force; the lower limit of the distribution and probability density functions of the outflow-to-inflow ratio is

$$\frac{\delta_{4\%} v_{4\%}^{72}}{1 - v_{4\%}^{72}} = 0.002476$$

The transformation of to the variable of the outflow-inflow ratio P is

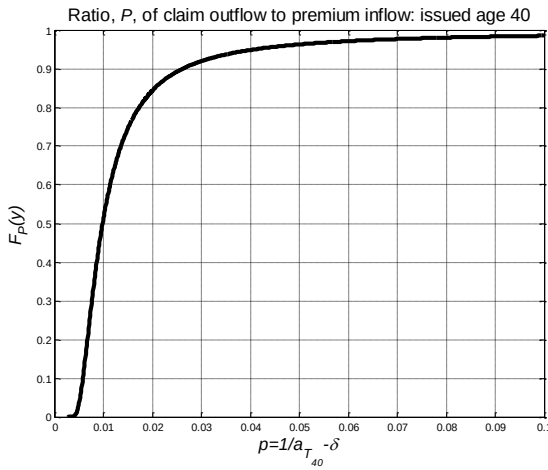


The distribution function of the ratio P for log-normal mortality is

$$F_P(p) = \frac{\Phi\left([\log(72 - \delta_{4\%}^{-1} \log(1 + \delta_{4\%} / p)) - 3.4085] / 0.3685\right)}{\Phi\left([\log 72 - 3.4085] / 0.3685\right)}$$

$$p \in (0.002476, \infty)$$

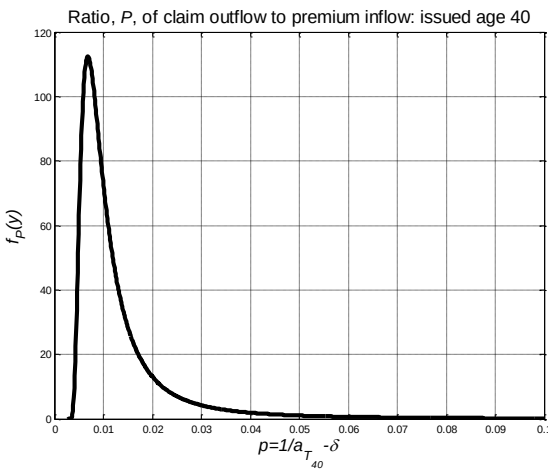
Note that because of the asymptote corresponding to $T_x = 0$, or $p = \infty$, it is easier and more practical to consider a restricted domain for the ratio. The graphics of the density and cumulative density of the ratio for log-normal mortality are graphed over 0 to 0.10 for the mortality parameters (of life-force).



The probability density functions of the ratio P for log-normal mortality is

$$f_P(p) = \frac{1}{p(p + \delta_{4\%})} \frac{\phi\left([\log(72 - \delta_{4\%}^{-1} \log(1 + \delta_{4\%} / p)) - 3.4085] / 0.3685\right)}{0.3685(72 - \delta_{4\%}^{-1} \log(1 + \delta_{4\%} / p)) \Phi\left([\log 72 - 3.4085] / 0.3685\right)}$$

$$\text{for } p \in (0.002476, \infty)$$



Notice that the direction of the skew is in the same direction as the claim payment function, that of the assurance, but it is more concentrated (the peak is sharper and more localized). The probability mass function is right skewed with relation to the outgo-to-inflow ratio, but since the transformation is a decreasing function of the future lifetime random

variable (past age 40), the probability distribution is left skew with relation to the age at which the claim is made and the premiums cease, i.e., the ratio is greater for an earlier the claim for which there is less premium earned to pay for the claim.

2.3.3. Bounds, P

For confidence intervals for the ratio, solving

$$0.95 = \Pr(lb < P < ub)$$

which is an equation in percentiles, with the specification of equal weight in either tail,

$$0.025 = \Pr(P < lb) = F_P(lb)$$

$$0.975 = \Pr(P < ub) = F_P(ub)$$

gives the bounds for the ratio of unit claim outgo to unit premium inflow,

$$LB(P, 95\%) = \frac{\delta v^{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.025 S_{T_0}(x)\right)\right)}}{1 - v}$$

$$UB(P, 95\%) = \frac{\delta v^{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.975 S_{T_0}(x)\right)\right)}}{1 - v}$$

which are each logistic functions of the 2.5th and the 97.5th percentiles of the future lifetime variable T_x . For the whole life insurance of a life aged 40 with continuous premiums payable while the cover is in force, the bounds for the ratio, of which the net premium is the ratio of the means, is

$$LB(P, 95\%) = \frac{\delta v^{72 - \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.025 S_{T_0}(40)\right)\right)}}{1 - v} = 0.004625$$

$$UB(P, 95\%) = \frac{\delta v^{72 - \exp\left(3.4085 + 0.36855 \Phi^{-1}\left(0.975 S_{T_0}(40)\right)\right)}}{1 - v} = 0.060183$$

$$\text{where } S_{T_0}(40) = \Phi\left([\log 72 - 3.4085] / 0.3685\right) = 0.99076$$

These intervals can also be derived from the confidence intervals on the insurance and annuity functions, and directly from the relevant percentiles of the future lifetime variable. The confidence interval for A was [\$0.105487, \$0.605442] and the confidence interval for a was [\$10.0599, \$22.8072]. The corresponding intervals for P , as just calculated, are

$$[0.105487 / 22.8072, 0.605442 / 10.0599] = [0.004625, 0.060183].$$

The net premium of $\bar{P}_{40}(\bar{A}_{40}) = \0.011829 per unit sum insured is much closer to the lower bound of the interval of \$0.004853 because of the right skew.

Given that the log-normal theory holds, an insurer writing this policy,

- There is a 2.5% probability that the present value of the claim payment to unit rate of premium receivable continuously will exceed \$0.060183.
- There is a 2.5% probability that the present value of the claim payment to unit rate of premium receivable continuously will be less than \$0.004625.

The probability that the present value of the claim payment to unit rate of premium receivable continuously, will not exceed the net premium income is

$$\begin{aligned} F_P(\bar{P}_{40}(\bar{A}_{40})) &= F_P(0.011829) \\ &= \frac{\Phi\left([\log(72 - \delta_{4\%}^{-1} \log(1 + \delta_{4\%} / 0.011829)) - 3.4085] / 0.3685\right)}{\Phi\left([\log 72 - 3.4085] / 0.3685\right)} \\ &= S_{T_{40}}\left(\delta_{4\%}^{-1} \log(1 + \delta_{4\%} / 0.011829)\right) \\ &= 0.652764 \end{aligned}$$

That is, there is a 65.3% probability that the net premium will be sufficient to cover the claim outgo in its entirety [since the actuarial method is

conservative] and a 34.7% probability that the net premium will be insufficient. The expression for the median of the distribution of the ratio is

$$\text{Median}(P) = \frac{\delta v^{\text{Median}(T)}}{1 - v^{\text{Median}(T)}} = \frac{\delta v^{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.50 S_{T_0}(x)\right)\right)}}{1 - v^{\omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left(0.50 S_{T_0}(x)\right)\right)}}$$

For the whole life insurance and continuous premiums at 4% to the life aged 40, given that the median future lifetime at age 40 is 41.91 further years, the median of the ratio is

$$\text{Median}(P) = \frac{\delta v^{41.9094}}{1 - v^{41.9094}} = 0.009396$$

per dollar sum of insurance. This is somewhat smaller than the net premium of \$0.011829. There is a 15.3% probability that the annual net cost, in present value terms, will lie between the distributional median and the net annual level premium. This demonstrates that the calculation of the net level premium by the actuarial method is conservative (anywhere just before the mode of lifetime at birth), since the mean is larger than the median and the mode for ages before the mode of the future lifetime variable. Note that after the mode of T_0 [mid 80's] the actuarial method based on means ceases to be conservative and underestimates **50% likely** net costs as well as **most likely** net costs).

2.4. Insurance Reserve, V , with log-normal mortality

Premium income for a traditional policy for insured (x) is sufficient to cover the claim whenever the insured lives a further t years

$$\begin{aligned} P(1 - v^t) / \delta &= S v^t \\ v^{-t} - 1 &= \delta S / P \\ v^{-t} &= 1 + \delta S / P \\ -t \log v &= \log(1 + \delta S / P) \\ t &= \frac{\log(1 + \delta S / P)}{\delta} \end{aligned}$$

If the insured does not live $(1 / \delta) \log(1 + \delta S / P)$ years after taking out the policy, the total net premium will be insufficient to cover the claim. If the insured lives beyond this age, the total net premium will be in excess of that required to pay the claim. A reserving system should ensure that **(1)** the premium for a specific policy be kept distinctly matched and saved against the claim for which the premium is received (that is, take into account saved premium) and **(2)** average the deficiency and surplus of premium over policies to reduce the statistical variation around matching premium to claims. That is, all premium income from a policy should be saved to fund the claim and **i)** a surplus for a late claim should fund deficiencies for short lived lives while **ii)** a deficiency from an early claim should be funded by long lived lives. A terminal reserve keeps premium of a policy matched to the claim but does not account for saved premium to fund the claim and, for this reason, the reserve becomes excessively large. It does not take into account saved premium and is the reserve required if all premiums are spent or are not kept matched against the specific policy for which the premiums were paid to fund. For example, near the limit of lifespan, the net terminal reserve approaches unity; the reserve then is that for the complete sum insured but takes no account of the premiums paid throughout the insured's lifetime to fund the claim. With these reservations in mind, the terminal reserve is now considered for the whole of life policy for a life aged 40.

2.4.1. Moments, V

The reserve to be held to fund future net outgo arising from an assurance policy is the present value of the claim arising from the policy less the present value of the future premium income received for the policy. For a traditional whole life policy to a life aged x at issue t years ago, with premiums payable continuously at rate P per annum while the policy is in force, the net prospective reserve is

$${}_t V_x = v^{T_{x+t}} - P \bar{a}_{T_{x+t}} = (1 + P / \delta) v^{T_{x+t}} - P / \delta$$

It is emphasized again that this does not account for the t years of earned premium that should be saved and matched against the claim when it occurs. This variable V has the same distributional shape, to within a change of scale and location, as the variables underlying A and a since each are linear functions of v^T . Seeing that the net prospective reserve is a linear function of the life contingent present value factor, the moments of the net prospective reserve can be derived from the corresponding mo-

ments of the insurance functions, in a similar manner as the annuity function, but in t years ahead of the policy inception at age x . The first raw moment of the random variable for the reserve to be held in terms of future net outflow of funds is

$$E({}_t V_x) = (1 + P / \delta) E(v^{T_{x+t}}) - P / \delta = (1 + P / \delta) \bar{A}_{x+t} - P / \delta$$

The second raw moment is

$$\begin{aligned} E({}_t V_x^2) &= E\left(\left((1 + P / \delta) v^{T_{x+t}} - P / \delta\right)^2\right) \\ &= (P / \delta)^2 E\left(\left((\delta / P + 1) v^{T_{x+t}} - 1\right)^2\right) \\ &= (P / \delta)^2 E\left((\delta / P + 1)^2 v^{2T_{x+t}} - 2(\delta / P + 1) v^{T_{x+t}} + 1\right) \\ &= (P / \delta)^2 \left({}^2 \bar{A}_{x+t} (\delta / P + 1)^2 - 2 \bar{A}_{x+t} (\delta / P + 1) + 1\right) \end{aligned}$$

Similarly the third raw moment is

$$\begin{aligned} E({}_t V_x^3) &= E\left(\left((1 + P / \delta) v^{T_{x+t}} - P / \delta\right)^3\right) \\ &= (P / \delta)^3 \left({}^3 \bar{A}_{x+t} (\delta / P + 1)^3 - 3 {}^2 \bar{A}_{x+t} (\delta / P + 1)^2 + 3 \bar{A}_{x+t} (\delta / P + 1) - 1\right) \end{aligned}$$

and the general k^{th} raw moment is

$$\begin{aligned} E({}_t V_x^k) &= E\left(\left((1 + P / \delta) v^{T_{x+t}} - P / \delta\right)^k\right) \\ &= (P / \delta)^k \sum_{r=0}^k (-1)^{k-r} \times {}^k C_r {}^r \bar{A}_{x+t} (\delta / P + 1)^r \end{aligned}$$

For the whole life policy with continuous payments of premium whilst the policy is in force, issued to a life aged 40 at inception, with money worth 4% per annum flat, the log-normal net premium was

$$\bar{P}_{40}(\bar{A}_{40}) = 0.011829$$

and the net prospective reserve 10 years into the policy for the log-normal curve was

$$\begin{aligned} {}_{10} \bar{V}_{40} &= 1 - \left(\delta + \bar{P}_{40}(\bar{A}_{40})\right) \bar{a}_{50} \\ &= 0.1264634 \end{aligned}$$

which is the first moment of the random variable of the reserve. The further raw moments of the insurance function for the reserve moment calculations are, from either the formulae for the A function or by numerical integration

$$\begin{aligned} {}^k \bar{A}_{50} &= E(v^{k T_{50}}) \\ &= \int_0^{\omega-50} v^{kt} f_{T_{50}}(t) dt \\ &= \int_0^{62} v_{4\%}^{kt} \frac{\phi([\log(62-t) - 3.685] / 0.3685)}{0.3685(62-y) \Phi([\log 62 - 3.685] / 0.3685)} dt \end{aligned}$$

k	${}^k \bar{A}_{50} = E(v^{k T_{50}})$
1	0.328877
2	0.132378
3	0.065480
4	0.038670
5	0.026075
6	0.019274

Noting that

$$\delta / P = \log(1.04) / 0.011829 = 3.31565$$

The second raw moment is,

$$\begin{aligned} E({}_t V_x^2) &= (P / \delta)^2 \left({}^2 \bar{A}_{x+t} (\delta / P + 1)^2 - 2 \bar{A}_{x+t} (\delta / P + 1) + 1\right) \\ &= \frac{0.132378(4.31565)^2 - 2(0.328877)(4.31565) + 1}{(3.31565)^2} \\ &= 0.057023 \end{aligned}$$

The variance is then

$$\begin{aligned} \text{Var}({}_tV_x^2) &= E({}_tV_x^2) - E({}_tV_x)^2 \\ &= 0.057023 - (0.126466)^2 \\ &= 0.041029 \end{aligned}$$

and the standard deviation is

$$SD({}_tV_x^2) = 0.20256$$

Obviously, like the earlier functions, simple addition and subtraction of standard deviation does not result in a sensible probability interval because of the presence of skew and non-zero higher moments, although it is of interest that zero is less than one standard deviation (two thirds of a standard deviation) below the mean. Considering third order moments, the raw moment is

$$\begin{aligned} E({}_{10}V_{40}^3) &= (P/\delta)^3 \left({}^3\bar{A}_{50}(\delta/P+1)^3 - 3{}^2\bar{A}_{50}(\delta/P+1)^2 + 3\bar{A}_{50}(\delta/P+1) - 1 \right) \\ &= \frac{0.06548(4.3157)^3 - 3(0.13238)(4.3157)^2 + 3(0.32888)(4.3157) - 1}{(3.3157)^3} \\ &= 0.030852 \end{aligned}$$

The third central moment is

$$\begin{aligned} \text{Skew}({}_{10}V_{40}) &= E({}_{10}V_{40}^3) - 3E({}_{10}V_{40}^2)E({}_{10}V_{40}) + 2E({}_{10}V_{40})^3 \\ &= 0.030852 - 3(0.057023)(0.126466) + 2(0.126466)^3 \\ &= 0.013263 \end{aligned}$$

The cube root of the skewness is

$$\sqrt[3]{\text{Skew}({}_{10}V_{40})} = 0.236709$$

This value is larger than that of the standard deviation by around 35 cents in the dollar, which emphasizes the importance of not ignoring the third order measure of shape. The value of skewness, normalized for spread, is

$$\begin{aligned} \zeta_3({}_{10}V_{40}) &= \text{Skew}({}_{10}V_{40}) / SD({}_{10}V_{40})^3 \\ &= 0.013263 / 0.20256^3 \\ &= 1.59595 \end{aligned}$$

and the cube root, the third order measure of shape, is

$$\sqrt[3]{\zeta_3({}_{10}V_{40})} = 1.6862$$

The latter value in excess of 1 indicates that the distributional skewness dominates the spread. The fact that the distribution is right skew, which indicates that the weighted average will be pulled away from the mode in the direction of the skew, means that the other measures of location will lie near zero (between 0.1264 and zero). The median of this distribution is

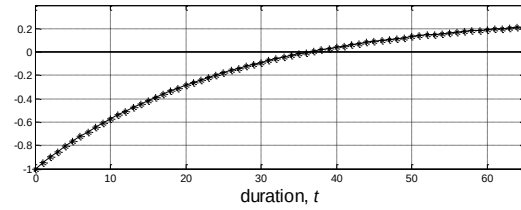
$$\begin{aligned} \text{Median}({}_{10}V_{40}) &= (1 + P/\delta)v^{\text{Median}(T_{50})} - P/\delta \\ &= (1.3016)^{62 - \exp(3.4085 + 0.36855\Phi^{-1}(0.975S_{T_0}(50)))} - 0.3016 \\ &= 0.06746 \end{aligned}$$

which is almost half the mean of 0.1264. Early claims are the main solvency concern of this form of policy due to the discontinuation of ongoing premium income after the claim is made, to the extent to which the total premium inflow does not cover the claim. That is, early claims are loss making. This is when the terminal reserve is at its smallest (when individual losses are greatest).

A policy of this type that does not have a margin for profit and contingencies only becomes profitable for the insurer if the insured lives to age 77.3; that is, a duration of 37.3 years

$$\begin{aligned} T_x &= \frac{\log(1 + \delta a_x / A_x)}{\delta} \\ &= \frac{\log(1 + (\log 1.04) \bar{a}_{40} / \bar{A}_{40})}{\log 1.04} \\ &= 37.3 \end{aligned}$$

immediately after which the continuous annual premium first exceeds the insurance benefit. It is mandatory that the benefit reserve is viewed in the context of saved premium.



The above graph shows the present value of premium less the present value of the claim. If a claim is made immediately at age 40 (although unlikely), the policy immediately incurs a loss of 1 without earning any premium (when the terminal reserve is, a little ironically, zero). The curve is -1 at this point. After a duration of 37.3 years (10 years is being considered in the example), the curve intersects the horizontal axis and the policy premium exceeds the claim and from which point the policy becomes profitable.

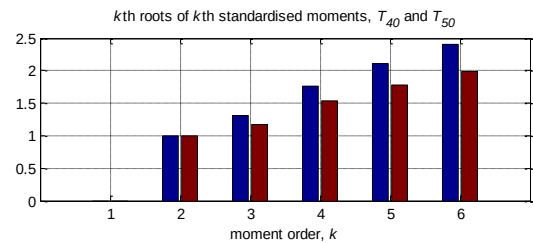
The further indices of shape, the k^{th} central moments and normalized k^{th} roots of the k^{th} central moments are tabulated in the table below for the 10 year duration and have been calculated by integration with respect to the log-normal density

$$\begin{aligned} &E\left(\left((1 + P/\delta)v^{T_{50}} - P/\delta - {}_{10}\bar{V}_{40}\right)^k\right) \\ &= \int_0^{62} \left((1 + P/\delta)v^t - P/\delta - {}_{10}\bar{V}_{40}\right)^k \frac{\phi(\log(62-t) - 3.685)/0.3685}{0.3685(62-y)\Phi(\log 62 - 3.685/0.3685)} dt \\ &= \int_0^{62} \left((1.3016)v^t - 0.42806\right)^k \frac{\phi(\log(62-t) - 3.685)/0.3685}{0.3685(62-y)\Phi(\log 62 - 3.685/0.3685)} dt \end{aligned}$$

Central moments and k^{th} roots of normalized central moments

k	Central moments of V	$\sqrt[k]{\zeta_k({}_{10}V_{40})}$
1	0	0
2	0.041029	1
3	0.013263	1.16862
4	0.009592	1.54502
5	0.006020	1.77571
6	0.004272	1.98864

Notice that these numbers are different, although similar, to the indices of shape for the distributions of insurance and annuity. They are similar because the random variable is based on the present value factor as a function of lifetime while they are different because the lifetime random variable is now conditional on survival to age 50 rather than 40.



Displayed in the bar graph is the normalized central moments based on the mortality variable at ages 40 (T_{40} , in blue) and 50 (T_{50} , in maroon) for comparison with the table of moments for those of the annuity and insurance at age 40 on pages 10 and 12 (which pertain to T_{50}). Notice how the distribution is evolving with age. The indices of shape in the right hand column in the tables for the k^{th} roots of the standardized k^{th} moments (as plotted in the bar graph), while still increasing monotonically, the rate of increase is smaller for the distribution based on the T_{50} period present value factor than those of the T_{40} period present value factor (the bars are smaller except for the standardized variance of 1). The degree of skewness, kurtosis, and higher order attributes of shape become more subdued as age progresses. There is a change in sign of the odd moments past $\text{Mode}(T_0)$ [a table of numbers and plot is omitted], i.e., for a terminal reserve for this policy past a duration of about 35 years.

2.4.2. Distribution, V

Letting V be the parametric distribution for the insurance reserve for a given premium

$$V = (1 + P/\delta)v^{T_{x+t}} - P/\delta$$

which is the amount that has to be held to cover future net claims (not including earned premium) on the policy. The domain for V is

$$0 < T_{x+t} < \infty$$

$$0 < v^{T_{x+t}} < 1$$

$$0 < (1 + P/\delta)v^{T_{x+t}} < (1 + P/\delta)$$

$$-P/\delta < L < 1$$

Intuitively, the largest loss that the writer can sustain on the policy is if there is a claim immediately at the considered duration, in which case the loss is 1. The smallest possible loss is if the life lives forever, in which case, the loss is a negative perpetuity, $-P/\delta$. Assuming that there is a limit on lifespan and the limit to future lifetime is $\omega - x$ as per the log-normal theory, the smallest possible loss on the policy is

$$(1 + P/\delta)v^{\omega-x-t} - P/\delta$$

so the domain becomes

$$(1 + P/\delta)v^{\omega-x-t} - P/\delta < V < 1$$

By the technique of distribution functions to transform the random variable,

$$\begin{aligned} F_V(y) &= \Pr(V \leq y) \\ &= \Pr\left((1 + P/\delta)v^{T_{x+t}} - P/\delta \leq y\right) \\ &= \Pr\left(v^{T_{x+t}} \leq (y + P/\delta)/(1 + P/\delta)\right) \\ &= \Pr\left(T_{x+t} \log v \leq \log((y + P/\delta)/(1 + P/\delta))\right) \\ &= \Pr\left(T_{x+t} \geq \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \\ &= S_{T_{x+t}}\left(\frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \\ &= \frac{S_{T_0}\left(x + t + \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right)}{S_{T_0}(x + t)} \end{aligned}$$

In terms of traditional actuarial notation,

$$F_V(y) = {}_w p_{x+t} \quad \text{where } w = \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}$$

$$= {}_{w+x+t} p_0 / {}_{x+t} p_0$$

Differentiating the distribution function for the probability density function,

$$\begin{aligned} f_V(y) &= dF_V(y)/dy \\ &= \frac{d}{dy} S_{T_{x+t}}\left(\frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \\ &= \frac{d}{du} S_{T_{x+t}}(u) \times \frac{d}{dy} \left(\frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \\ &= \frac{1}{P + \delta y} f_{T_{x+t}}\left(\frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \\ &= \frac{1}{P + \delta y} f_{T_0}\left(x + t + \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) / S_{T_0}(x + t) \end{aligned}$$

which, in standard actuarial notation, is expressible as

$$f_P(p) = \frac{{}_w p_{x+t} \mu_{x+t+w}}{P + \delta y} = \frac{{}_{x+t+w} p_0 \mu_{x+t+w} / {}_{x+t} p_0}{P + \delta y}$$

where $w = \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}$ and $1/(P + \delta y)$ where is the Jacobian.

For the log-normal theory of mortality, the distribution function and probability density function are, respectively,

$$F_V(y) = \frac{\Phi\left(\left(\log\left(\omega - x - t - \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) - \mu\right) / \sigma\right)}{\Phi\left((\log(\omega - x - t) - \mu) / \sigma\right)}$$

$$y \in \left((1 + P/\delta)v^{\omega-x-t} - P/\delta, 1\right)$$

and

$$f_V(y) = \frac{1}{P + \delta y} \frac{\phi\left(\left(\log\left(\omega - x - t - \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) - \mu\right) / \sigma\right)}{\sigma \left(\omega - x - t - \frac{1}{\delta} \log \frac{1 + P/\delta}{y + P/\delta}\right) \Phi\left((\log(\omega - x - t) - \mu) / \sigma\right)}$$

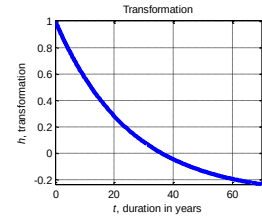
$$y \in \left((1 + P/\delta)v^{\omega-x-t} - P/\delta, 1\right)$$

For the whole of life policy for a life aged 40 at inception, paying level annual continuous premiums of \$0.01183, the lower limit of the probability functions of the net reserve 10 years into the policy occurs if there is a claim immediately, i.e., if a claim occurs at age $x + t = 50$, $T_{50} = 0$, $\omega - x - t = 62$ years away from the limit to lifespan

$$(1 + P/\delta)v^{\omega-x-t} - P/\delta = 1.3016v_{4\%}^{62} - 0.3016 = -0.1872$$

The transformation to the variable of the stochastic reserve V is

$$V = (1 + P/\delta)v^t - P/\delta$$

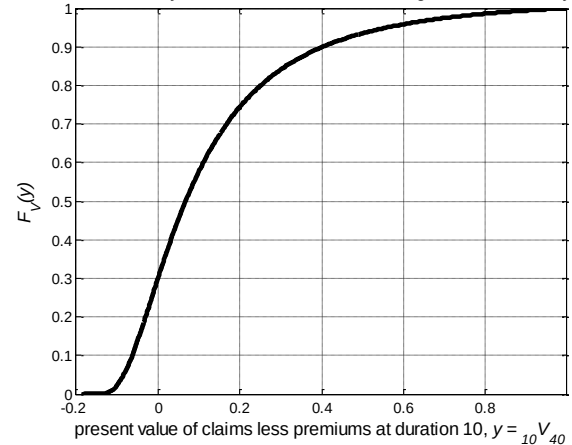


The distribution function of the reserve V with log-normal mortality is

$$F_V(y) = \frac{\Phi\left(\left(\log\left(62 - \frac{1}{\delta_{4\%}} \log \frac{1.3016}{y + 0.3016}\right) - 3.4085\right) / 0.3685\right)}{\Phi\left((\log(62) - 3.4085) / 0.3685\right)}$$

$$y \in (-0.1872, 1)$$

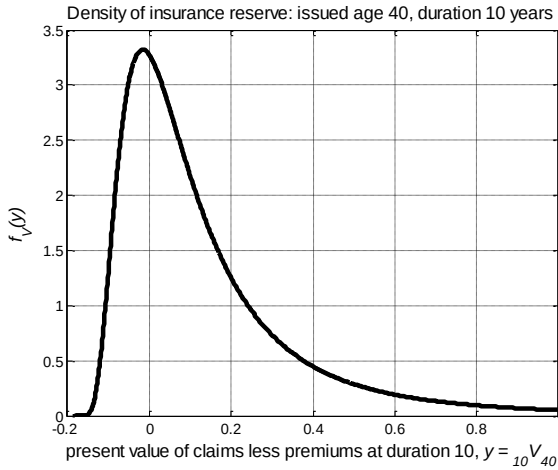
Cumulative density of insurance reserve: issued age 40, duration 10 years



and the probability density function of the reserve V for log-normal mortality is

$$f_V(y) = \frac{1}{0.01183 + \delta_{4\%} y} \times \frac{\phi\left(\left(\log\left(62 - \frac{1}{\delta_{4\%}} \log \frac{1.3016}{y + 0.3016}\right) - 3.4085\right) / 0.3685\right)}{0.3685 \left(62 - \frac{1}{\delta_{4\%}} \log \frac{1.3016}{y + 0.3016}\right) \Phi\left((\log(62) - 3.4085) / 0.3685\right)}$$

$y \in (-0.1872, 1)$



2.4.3. Bounds, V

Deriving confidence intervals for the reserve, by solving the probability equation for the tail percentiles

$$0.95 = \Pr(lb < L < ub)$$

with the specification of equal weight in either tail,

$$0.025 = F_V(LB)$$

$$0.975 = F_V(ub)$$

which yields the formulae

$$LB(P, 95\%) = (1 + P/\delta) v^{\omega - x - t - \exp\left(\mu + \sigma \Phi^{-1}\left(0.025 S_{T_0}(x+t)\right)\right)} - P/\delta$$

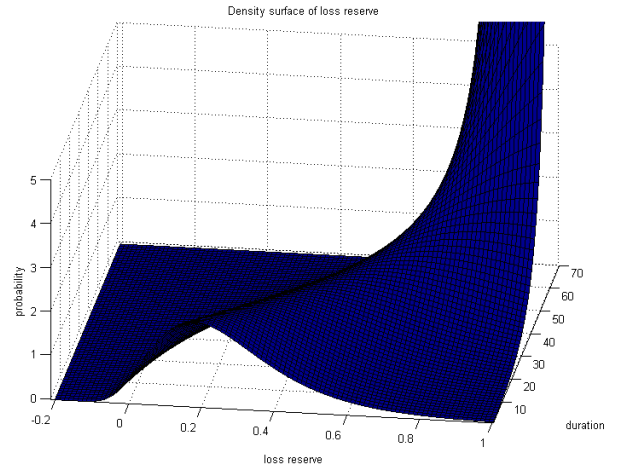
$$UB(P, 95\%) = (1 + P/\delta) \delta v^{\omega - x - t - \exp\left(\mu + \sigma \Phi^{-1}\left(0.975 S_{T_0}(x+t)\right)\right)} - P/\delta$$

which is the transformation of V evaluated at the 2.5th and 97.5th percentiles of the future lifetime variable at age $x + t$. For the whole life policy for a life aged 40 with premiums payable continuously whilst the policy is in force, 95% confidence bounds for the prospective reserve with duration 10 years into the policy, are

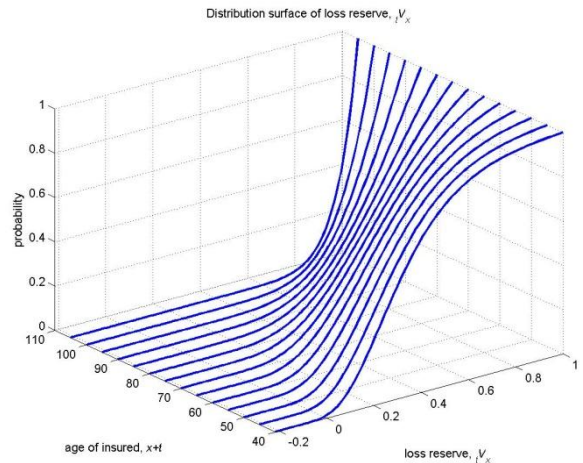
$$LB(P, 95\%) = (1.3016) v^{62 - \exp\left(3.403 + 0.3705 \Phi^{-1}\left(0.025 S_{T_0}(62)\right)\right)} - 0.3016 = -0.0987$$

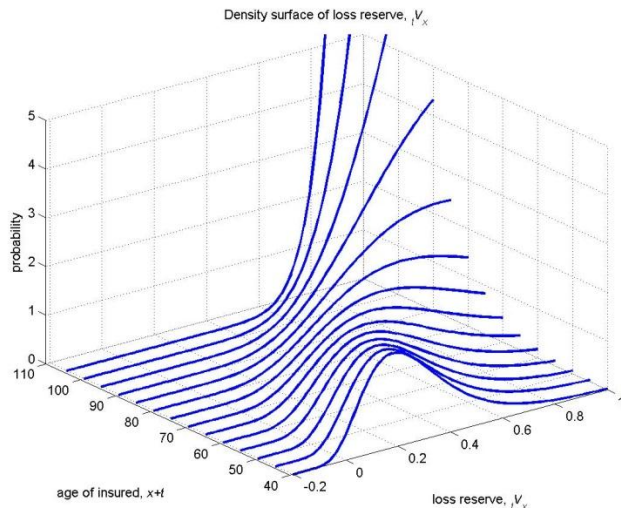
$$UB(P, 95\%) = (1.3016) v^{62 - \exp\left(3.403 + 0.3705 \Phi^{-1}\left(0.975 S_{T_0}(62)\right)\right)} - 0.3016 = 0.7038$$

noting that the 2.5th and 97.5th percentiles of the future lifetime variable at for a life aged 50 by the log-normal curve are 47.38 years and 6.58 years respectively; i.e., a 97.5% probability of a 50 year old living to 56.58 years of age and a 2.5% probability of living to 97.38 years. For the reserve, there is a 2.5% chance that a reserve of \$0.7038 will be insufficient, after allowing for future premium income, to cover the claim. On the other hand there is a 2.5% chance that there will be a profit or release from surplus of \$0.0987 or more (not including earned premium from age 40 to age 50).

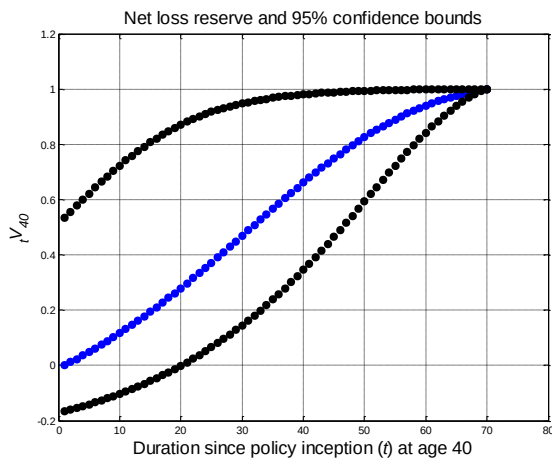


Letting the duration since the inception of the policy vary, the reserve function becomes a function of two arguments, one a stochastic variable and the other a conditioning variable. One can see how the benefit reserve required to back the net future cash flows evolves throughout the policies lifetime, given that the insured survives the specific duration since obtaining the cover at age 40. The modal peak and the mean net benefit reserve, which lies slightly to the right of the modal peak, are gradually migrating outwards towards unity with increasing modal height signifying increasing certainty as to when the claim will be made and premium will cease. This process continues whereby the mass of the density approaches a reserve of one and whereby the insured, if living, approaches the terminal age whereby a claim will immediately be made and premium will instantaneously cease. This is a convergence to a Dirac delta function in the density function as it asymptotes at the limit of lifespan. The distribution function rectangularizes to a step function as curvature is slowly removed as the probability mass increasingly concentrates into the mode until it has zero width and infinite height so that the distribution function reaches unity at the terminal age. This process can be more easily seen in the probability density and distribution functions with adjacent curves at five year duration intervals.





Such functions, in conjunction with simulation, will allow an insurer to assess different outcomes and how the experience will eventuate from different perspectives – most relevantly, likely **frequencies, costs, and chances**. This will give an actuary deeper scope for understanding the stochastic nature of the cash flows and associated reserves in conjunction with earned and saved premium. From this perspective and theoretical underpinning, the major actuarial function of analysis of profit whereby the realised experience is compared with what was assumed in the valuation basis can be compared and any discrepancies can be identified and used to update the valuation basis for future conditions. The most desirable feature of the analysis of profit conducted in a way consistent with the log-normal theory of mortality and a full distributional analysis, is that significance levels can be calculated and inferential hypotheses tested with a firm statistical basis to supplement the purely descriptive methods based on means.



The last diagram shows a series for the net reserve against duration since policy inception. Around the reserve are the 2.5% probability bands.

This brings this document to the last topic, the simulation of lifetimes, for which the task of comparing a valuation model and reserve system to actual outcomes can be analysed in advance and simulated for a Monte-Carlo distribution of a complete book of insurance and annuity business.

Part 3: Simulation of lifetime and life contingent cash flows by reference to the log-normal theory of mortality

3.0. Algorithm and discussion

The distribution function represents the cumulative probability of a life dying before a given value of age. The ordinate of this function ranges from zero to one, and for any given life, this vertical aspect is uniformly distributed between zero and one for a random realization of the argument age. To simulate a random lifetime, one can start with a random decimal representing the realized value of the distribution function, and then check which value of age this realization corresponds to. In functional terms, this is an inversion of the percentile of the distribution function. From the cumulative density function of the log-normal lifetime function

$$F_{T_x}(y) = {}_yq_x = 1 - \frac{S_{T_0}(x+y)}{S_{T_0}(x)}$$

Letting

$$F_{T_x} = u \sim U(0,1)$$

so that

$$u = 1 - \frac{S_{T_0}(x+y)}{S_{T_0}(x)}$$

$$\frac{S_{T_0}(x+y)}{S_{T_0}(x)} = 1 - u$$

$$S_{T_0}(x+y) = (1-u)S_{T_0}(x)$$

Remembering that

$$S_{T_0}(x+y) = \Phi([\log(\omega-x-y)-\mu]/\sigma)$$

and

$$S_{T_0}(x) = \Phi([\log(\omega-x)-\mu]/\sigma)$$

so that

$$\Phi([\log(\omega-x-y)-\mu]/\sigma) = (1-u)S_{T_0}(x)$$

and upon inversion of the cumulative unit normal density

$$[\log(\omega-x-y)-\mu]/\sigma = \Phi^{-1}((1-u)S_{T_0}(x))$$

$$\log(\omega-x-y) = \mu + \sigma \Phi^{-1}((1-u)S_{T_0}(x))$$

$$\omega-x-y = \exp\left(\mu + \sigma \Phi^{-1}((1-u)S_{T_0}(x))\right)$$

$$x+y = \omega - \exp\left(\mu + \sigma \Phi^{-1}((1-u)S_{T_0}(x))\right)$$

the simulated future lifetime of a life now aged x , T_x , is

$$y = \omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left((1-u)\Phi\left(\frac{\log(\omega-x)-\mu}{\sigma}\right)\right)\right)$$

That is, the simulated attained age $x+y$ for a life now aged x is

$$x+y = \omega - \exp\left(\mu + \sigma \Phi^{-1}\left((1-u)\Phi\left(\frac{\log(\omega-x)-\mu}{\sigma}\right)\right)\right)$$

Because of the symmetry of a unit uniform about 0.5, U has the same distribution as $1-U$ and the simulation formula can be written in a slightly simpler form as

$$x+y = \omega - \exp\left(\mu + \sigma \Phi^{-1}\left(u\Phi\left(\frac{\log(\omega-x)-\mu}{\sigma}\right)\right)\right)$$

Given the accuracy of the mortality parameters, the simulation algorithm based on the log-normal normal theory of mortality (for biological mortality) will be **accurate for any age past infant mortality and the sig-**

nificant range of non-biological mortality. This is essentially a formula for the (reflected) log-normal percentile, conditional to age x , where the $100(1-u)^{\text{th}}$ percentile is determined by a unit uniform U .

3.1. Simulation of 5 future lifetimes of 40 year old males

Five simulated unit uniform $u \sim U(0,1)$ variables are

$$\underline{u} = [0.5822, 0.9279, 0.5298, 0.6197, 0.0890]^T$$

In the language of my PhD thesis (almost entirely analogous to how a table is referred to by actuaries for valuations), the vector $\underline{u}$ forms a **basis** for population mortality, in this case, for the five lives aged 40. This vector of unit uniform random variates will be used in the following illustrations. Inserting these into the simulation algorithm gives the vector of five simulated future lifetimes of five lives who have reached age 40

$$\underline{T}_{40} = [44.1077, 54.3896, 42.7207, 44.0824, 23.2716]$$

or in tabular form

life, j	simulated future lifespans	
	T_{40}	$40 + T_{40}$
1	44.1077	84.1077
2	54.3896	94.3896
3	42.7202	82.7202
4	44.0824	84.0824
5	23.2716	63.2716

For example, the first of the simulated lifespans was calculated from the first simulated unit uniform variate, a random number between 0 and 1, $u = 0.5822$

$$y = \omega - x - \exp\left(\mu + \sigma \Phi^{-1}\left((1-0.5822)\Phi\left(\frac{\log(\omega-x)-\mu}{\sigma}\right)\right)\right)$$

for the life now aged 40

$$y = \omega - 40 - \exp\left(\mu + \sigma \Phi^{-1}\left((1-0.5822)\Phi\left(\frac{\log(\omega-40)-\mu}{\sigma}\right)\right)\right)$$

according to the log-normal mortality parameters $\omega = 112$, $\mu = 3.40848$, and $\sigma = 0.368547$

$$y = 112 - 40 - \exp\left(3.4085 + 0.3685\Phi^{-1}\left(0.4178\Phi\left(\frac{\log(112-40)-3.4085}{0.3685}\right)\right)\right)$$

which evaluates for $T_{40} = 44.1078$ future life years. This is a total lifespan of 84 years 1 month and 9 days, i.e., 84 years last birthday conditional on survival to age 40. The four other simulated future lifetimes of the basis evaluate to 54.39 years, 42.72 years, 44.08 years, and 23.27 years as per the tabulated entries to the table. This makes simulating a lifespan as easy as flipping a coin and makes the simulation of insurance and pension systems a straightforward task. It is possible to do this directly from a life table although this is an awkward and time consuming numerical task. Given the goodness of the log-normal curve and mortality parameters, the difference is negligible, but the ease of application is not; this algorithm now makes simulation of mortality and mortality dependent cash flows a brilliant tool for both students and industry.

Students can utilize the simulation algorithm to get a feel for the mortality/lifespan variable and the nature of contingent cash flows, as well as a method by which learning bodies can ensure that their students have pre-experience in designing insurance and reinsurance systems, in a similar manner as pilots must have at least a certain number of hours of simulated flight time (simulated experience) before qualifying. This way, life contingent stochastic cash flows can become second nature to students before entering industry.

Industry can utilize the algorithm for Monte-Carlo simulations of insurance and annuity business as well as pretest proposed reinsurance programs. For example, to assess the degree to which the risk pooling of an insurer, as its *raison d'être*, meets its intended goals of reduced variation; in the context of profit margins, setting bonuses, and other design considerations such as ceding large aggregate right tail losses with reinsurance; if the study indicates that the design has not met its goals (with high

enough likelihood), the system can be redesigned including rebalancing and testing the ceded business reinsured, and then simulated (for a full distribution) again and reassessed in terms of actual verse expected.

3.2. Block of whole of life insurance for 5 lives aged 40

The expected value of the insurance is the A function

$$E(v^{T_{40}}) = \bar{A}_{40} = 0.231715$$

per dollar sum insured, while the actual cost of the insurance is

$$v^{T_{40}}$$

From the simulated lifetimes, the actual costs of the 5 policies are 0.1773, 0.1185, 0.1872, 0.1775, and 0.4014 per dollar of insurance, from which actual verse expected comparisons can be made;

life, j	Expected, $\bar{A}_{40}$	T_{40}	Actual, $v^{T_{40}}$	Deviance
1	0.231715	44.1077 years	0.177295854	0.0545
2	0.231715	54.3896 years	0.118457753	0.1133
3	0.231715	42.7202 years	0.18721142	0.0445
4	0.231715	44.0824 years	0.177471869	0.0542
5	0.231715	23.2716 years	0.401427334	-0.1697

The actual verse expected analysis for curtate functions, K_x can proceed by (1) converting the continuous A function to that which applies to benefits paid at the end of year of claim and (2) rounding the T_{40} variates up to the nearest integer for the simulated future timing for insurance paid at the end of year of death

$$E(v^{K_{40}+1}) = A_{40} = 0.22722 \text{ for comparisons with } v^{K_{40}+1}$$

where $K_{40} = \text{integer}(T_{40})$

life, j	Expected, A_{40}	$K_{40} + 1$	Actual, $v^{K_{40}+1}$	Deviance
1	0.227227	45 years	0.17119841	0.056028588
2	0.227227	55 years	0.11565551	0.111571487
3	0.227227	44 years	0.17804635	0.049180652
4	0.227227	45 years	0.17119841	0.056028588
5	0.227227	24 years	0.39012147	-0.162894474

3.3. Block of life annuities for 5 lives aged 40

The expected value of the stream of annuity payments is the a function

$$E((1-v^{T_{40}})/\delta) = \bar{a}_{40} = (1-\bar{A}_{40})/\delta = 19.5888$$

per dollar per annum. The actual cost of the payment stream is

$$(1-v^{T_{40}})/\delta$$

From the five simulated lifetimes, the actual costs of the five life annuities are 20.98, 22.48, 20.72, 20.97, and 15.26 per dollar of annual payment;

life, j	Expected, $\bar{a}_{40}$	T_{40}	Actual	Deviance
1	19.5888	44.1077 years	20.9763	-1.387
2	19.5888	54.3896 years	22.4764	-2.888
3	19.5888	42.7202 years	20.7235	-1.135
4	19.5888	44.0824 years	20.9718	-1.383
5	19.5888	23.2716 years	15.2616	4.327

For benefits paid continuously; to convert to immediate annuity or annuity in arrear, the continuous a is converted to the required discrete annuity and the simulated lifetime variates converted to the equivalent curtate variates. For example, for annuity in advance, the expected value is

$$E((1-v^{K_{40}})/\delta) = \ddot{a}_{40} = (1-A_{40})/\delta = 20.0921$$

and can be compared with actual

$$(1-v^{K_{40}})/\delta$$

where $K_{40} = \text{integer}(T_{40})$

life, j	Expected, $\ddot{a}_{40}$	K_{40}	Actual	Deviance
1	20.0921	44 years	20.9571317	-0.865
2	20.0921	54 years	22.4299406	-2.338
3	20.0921	43 years	20.7755477	-0.683
4	20.0921	44 years	20.9571317	-0.865
5	20.0921	23 years	15.1520362	4.940

Such a simulated system can proceed for all of the life contingent cash flows on the balance sheet by these same principles inclusive of limited terms, with and without endowments; say, for each of the assurance functions

$${}_nE_x \bar{A}_x \bar{A}_{x:n}^1 \bar{A}_{x:n}$$

and annuity functions

$$\bar{a}_x {}_m\bar{a}_x \bar{a}_{x:n} {}_m\bar{a}_{x:n}$$

on the balance sheet of an insurer (or pension provider), based on various rating factors and mortality parameters, by these same principles; and, if material, in conjunction with simulated rates of interest.

From bulk computation of a programming language, such lifetime simulation is also possible for an entire national population such as simulating forward from census data; a census mortality simulation based on the population pyramids for aggregate populations and sub-populations by various qualitative demographic variables.

The following simulation will consider five lives for various ages and various sums insured considering only the

$\bar{A}_x$ and $\bar{a}_x$ functions

for a simplified insurance system for the five policy holders.

3.4. Simulation of an insurance system with five policy holders

By this reasoning, a complete insurance system can be simulated easily. For a simple example, say that the five policyholders who are taking out policies are 35, 40, 42, 57, and 48 and that the policy holders pay premium continuously for a whole life insurance which is payable at the moment of death. The sums insured are, respectively, \$50,000, \$35,000, \$20,000, \$65,000, and \$40,000. The relevant insurance and annuity functions, from the log-normal curve, are

age, x	$\bar{A}_x$	$\bar{a}_x$
35	0.193072	20.5740267
40	0.231715	19.5887565
42	0.248963	19.1489889
57	0.413215	14.9610997
48	0.30723	17.6633708

The simulated aggregate claim process for the five policies is then

Simulation of insurance business							
Whole of life insurance, payable at the moment of death							
life, j	age, x	A_x	sum insured, S	expected, SA_x	future lifetime, $T(x)$	actual, $Sv^{T(x)}$	deviance, exp. vs actual
1	35	0.193072	\$50,000.00	\$9,653.60	49.0673 years	\$7,297.77	\$2,355.83
2	40	0.231715	\$35,000.00	\$8,110.03	54.3896 years	\$4,146.02	\$3,964.00
3	42	0.248963	\$20,000.00	\$4,979.26	40.7467 years	\$4,045.55	\$933.71
4	57	0.413215	\$65,000.00	\$26,858.98	28.5077 years	\$21,248.68	\$5,610.29
5	48	0.30723	\$40,000.00	\$12,289.20	16.3236 years	\$21,086.99	-\$8,797.79
				\$61,891.06		\$57,825.01	\$4,066.05

the lives, each with the same vector $\underline{u}$ for the mortality basis but with the different ages, produces claim payments in 49 years, 54 years, 41 years, 29 years, and 16 years. The present value of the insurance benefits is \$57,825.01 which is slightly less than the expected cost; a total of \$4,066.05 less than expected.

For the simulated aggregate premium process for the five policies,

Life annuity for wol insurance premiums, payable cotinuously							
life, j	age, x	a_x	annual premium, $P(A_x)$	expected, $P(A_x) a_x$	future lifetime, $T(x)$	actual, $P(1-v^{T(x)})/\delta$	deviance, exp. vs actual
1	35	20.57402671	\$469.21	\$9,653.60	49.0673 years	\$10,217.28	-\$563.68
2	40	19.58875651	\$414.01	\$8,110.03	54.3896 years	\$9,305.57	-\$1,195.54
3	42	19.14898888	\$260.03	\$4,979.26	40.7467 years	\$5,288.78	-\$309.52
4	57	14.96109971	\$1,795.25	\$26,858.98	28.5077 years	\$30,809.75	-\$3,950.78
5	48	17.66337081	\$695.74	\$12,289.20	16.3236 years	\$8,387.55	\$3,901.65
				\$61,891.06		\$64,008.93	-\$2,117.87

The **total expected premium** from the five policyholders is \$61,891.06, that which exactly covers the expected cost of the insurance policies. Given the actual lifespans of the policyholders, there is 49, 54, 41, 29, and 16 years worth of annual premium for each of the respective policyholders. This totals a present value of \$64,008.93. This is \$2,117.87 more than expected. In total, by the mortality basis of this simulation, the five policies based upon net premiums would have amounted to \$4,066.05 less claims paid and \$2,117.87 more premium received, a total of \$6,183.92. Another simulated mortality basis may well have led to a loss. A Monte-Carlo simulation of many, say 1,000 or 10,000 possible outcomes, would generate a net distribution for the aggregate experience of the five policies. Is this what the company who has written the business was expecting? Whether or not that it is, it is better to know beforehand! The one important point to this is that, for a realistic and prudent insurer, it is mandatory to save the premiums in order to pay claims and that well chosen stop-loss reinsurance may be an important consideration to the risk pooling goals of reduced variation in statistical outcome. Simulation may well become a vital tool for the design of the reinsurance programs of insurers working in collaboration with the reinsurers.

Bibliography

1. The Actuarial Education Company, ActEd, (2004) *Course notes for subject 104*. Faculty and Institute of Actuaries, Great Britain.
2. Andreeva, M (2007), About Mortality Data for Australia (updated version). Human Mortality Database (HMD). www.mortality.org
3. Everitt, B. S., (2002), *The Cambridge Dictionary of Statistics*, 2nd Edition. Cambridge University Press.
4. Human Mortality Database, www.mortality.org.
5. Klugman, S. A., Panjer, H. H., Willmot, G. E., (1998), *Loss Models*. New York: Wiley.
6. Lloyd-Jones, A., (2007) Actuarial Honours Thesis. Department of Actuarial Studies, Macquarie University.
7. McCullagh, P., Nelder, J. A., *Generalized Linear Models*, 2nd Edition. Chapman & Hall/CRC.
8. Neill, A. (1979), *Life Contingencies*. Heineman Publishing. London.
9. Nelson, W., (1982), *Applied Life Data Analysis*, New York: Wiley.
10. Parmenter, M. M., (1999) *Theory of Interest and Life Contingencies with Pension Applications: A Problem Solving Approach*, 3rd Edition. Actex Publications. Winsted, Connecticut.
11. Pollard, A. H. (1949). Methods of Forecasting Mortality using Australian Data. *Journal of the Institute of Actuaries*, 75, 151-182.
12. Pollard, J. H. (1995), On the Changing Shape of the Australian Mortality Curve. *Actuarial Studies and Demographic Research Paper Series*. Macquarie University.
13. Wackerly, D. D., W. Mendenhall III, and R. L. Scheaffer (2002), *Mathematical Statistics with Applications*, 6th Edition. Duxbury Advanced Series. Duxbury.
14. Wilmoth, J.R. Andreev, K. Jdanov, D. and Gleij D.A. with the assistance of Boe, C. Bubenheim, M. Philipov, D. Shkolnikov, V. Vachon1 P. (2007), *BRIEF SUMMARY of the Methods Protocol for the Human Mortality Database*. Human Mortality Database (HMD). www.mortality.org

APPENDIX A: LIFE TABLE

Human Mortality Database, 2004 Cross-Sectional Australian Males¹⁰

[x]	l_x	d_x	p_x	q_x	e_x	σ_x
0	100 000	523	.99477	.00523	78.07	15.31
1	99 477	50	.99950	.00050	77.48	14.26
2	99 427	20	.99980	.00020	76.52	14.16
3	99 407	28	.99972	.00028	75.53	14.12
4	99 380	16	.99984	.00016	74.55	14.07
5	99 364	20	.99980	.00020	73.56	14.04
6	99 344	9	.99991	.00009	72.58	14.00
7	99 335	13	.99987	.00013	71.59	13.98
8	99 322	13	.99987	.00013	70.60	13.96
9	99 309	10	.99990	.00010	69.60	13.94
10	99 299	13	.99987	.00013	68.61	13.92
11	99 286	11	.99989	.00011	67.62	13.90
12	99 275	20	.99980	.00020	66.63	13.88
13	99 255	12	.99988	.00012	65.64	13.85
14	99 243	19	.99981	.00019	64.65	13.83
15	99 225	23	.99977	.00023	63.66	13.81
16	99 202	27	.99973	.00027	62.68	13.77
17	99 175	45	.99955	.00045	61.69	13.74
18	99 130	68	.99931	.00069	60.72	13.68
19	99 062	82	.99917	.00083	59.76	13.59
20	98 980	83	.99916	.00084	58.81	13.49
21	98 897	96	.99903	.00097	57.86	13.38
22	98 801	71	.99928	.00072	56.92	13.27
23	98 730	71	.99928	.00072	55.96	13.18
24	98 658	89	.99910	.00090	55.00	13.10
25	98 570	82	.99917	.00083	54.05	13.00
26	98 488	100	.99898	.00102	53.09	12.92
27	98 387	84	.99915	.00085	52.15	12.81
28	98 304	98	.99900	.00100	51.19	12.72
29	98 205	97	.99901	.00099	50.24	12.63
30	98 108	109	.99889	.00111	49.29	12.53
31	97 999	119	.99879	.00121	48.35	12.43
32	97 881	115	.99882	.00118	47.41	12.33
33	97 765	125	.99872	.00128	46.46	12.23
34	97 640	100	.99898	.00102	45.52	12.12
35	97 541	98	.99900	.00100	44.57	12.04
36	97 443	111	.99886	.00114	43.61	11.96
37	97 332	106	.99891	.00109	42.66	11.88
38	97 226	115	.99882	.00118	41.71	11.80
39	97 111	138	.99858	.00142	40.76	11.72
40	96 973	144	.99852	.00148	39.82	11.63
41	96 830	137	.99858	.00142	38.88	11.53
42	96 692	147	.99848	.00152	37.93	11.45
43	96 545	188	.99805	.00195	36.99	11.36
44	96 357	194	.99799	.00201	36.06	11.25
45	96 163	179	.99814	.00186	35.13	11.15
46	95 984	229	.99761	.00239	34.20	11.05
47	95 755	228	.99762	.00238	33.28	10.94
48	95 527	239	.99750	.00250	32.36	10.83
49	95 288	277	.99709	.00291	31.44	10.72
50	95 011	285	.99700	.00300	30.53	10.60
51	94 726	332	.99650	.00350	29.63	10.48
52	94 394	316	.99665	.00335	28.73	10.35
53	94 078	352	.99626	.00374	27.83	10.24
54	93 726	416	.99556	.00444	26.93	10.11

[x]	l_x	d_x	p_x	q_x	e_x	σ_x
55	93 310	403	.99568	.00432	26.05	9.97
56	92 907	439	.99528	.00472	25.16	9.85
57	92 469	528	.99429	.00571	24.28	9.72
58	91 941	549	.99403	.00597	23.42	9.57
59	91 392	604	.99339	.00661	22.56	9.42
60	90 788	730	.99196	.00804	21.71	9.27
61	90 058	713	.99208	.00792	20.89	9.10
62	89 344	784	.99122	.00878	20.06	8.94
63	88 560	892	.98993	.01007	19.23	8.78
64	87 668	1 023	.98833	.01167	18.43	8.60
65	86 645	1 048	.98791	.01209	17.65	8.42
66	85 598	1 127	.98683	.01317	16.86	8.24
67	84 470	1 247	.98524	.01476	16.09	8.06
68	83 223	1 431	.98281	.01719	15.33	7.87
69	81 793	1 537	.98121	.01879	14.60	7.67
70	80 256	1 754	.97814	.02186	13.88	7.47
71	78 502	1 784	.97727	.02273	13.19	7.26
72	76 717	1 992	.97403	.02597	12.49	7.05
73	74 725	2 201	.97054	.02946	11.83	6.84
74	72 523	2 310	.96815	.03185	11.18	6.62
75	70 214	2 467	.96487	.03513	10.55	6.41
76	67 747	2 709	.96001	.03999	9.94	6.19
77	65 038	2 812	.95677	.04323	9.35	5.97
78	62 226	3 093	.95029	.04971	8.77	5.76
79	59 133	3 245	.94512	.05488	8.23	5.54
80	55 888	3 255	.94176	.05824	7.71	5.32
81	52 633	3 569	.93219	.06781	7.19	5.11
82	49 064	3 704	.92451	.07549	6.71	4.90
83	45 360	3 840	.91535	.08465	6.26	4.69
84	41 520	3 840	.90751	.09249	5.84	4.48
85	37 680	3 799	.89918	.10082	5.43	4.27
86	33 881	3 744	.88949	.11051	5.04	4.08
87	30 137	3 761	.87521	.12479	4.67	3.89
88	26 376	3 792	.85624	.14376	4.33	3.71
89	22 584	3 429	.84818	.15182	4.06	3.52
90	19 156	3 047	.84095	.15905	3.79	3.33
91	16 109	2 870	.82184	.17816	3.50	3.16
92	13 239	2 455	.81454	.18546	3.26	2.98
93	10 784	2 192	.79671	.20329	3.01	2.82
94	8 591	1 833	.78670	.21330	2.77	2.66
95	6 759	1 643	.75686	.24314	2.52	2.52
96	5 116	1 337	.73864	.26136	2.33	2.39
97	3 779	1 058	.71996	.28004	2.16	2.27
98	2 720	814	.70093	.29907	2.00	2.15
99	1 907	607	.68167	.31833	1.85	2.03
100	1 300	439	.66231	.33769	1.72	1.93
101	861	307	.64298	.35702	1.60	1.82
102	554	208	.62381	.37619	1.48	1.72
103	345	136	.60492	.39508	1.38	1.62
104	209	86	.58644	.41356	1.28	1.52
105	122	53	.56846	.43154	1.18	1.40
106	70	31	.55109	.44891	1.08	1.26
107	38	18	.53440	.46560	.95	1.08
108	21	10	.51847	.48153	.78	.83
109	11	5	.50333	.49667	.50	.50
110+	5	5	.00000	1.00000	.00	.00

¹⁰ All calculations are based on the q_x values to the stated five decimal places, the series direct from the Human Mortality Database.

APPENDIX B:

Survivorship Commutation Functions @ 4%

[x]	D_x	N_x	S_x	$\ddot{a}_x$	$Var(\Sigma v^y)$
0	100 000.00	2444 022.5	54098 534	24.4402	2.2965
1	95 650.96	2344 022.5	51654 511	24.5060	1.6474
2	91 926.09	2248 371.5	49310 489	24.4585	1.6276
3	88 372.80	2156 445.4	47062 117	24.4017	1.6587
4	84 950.05	2068 072.6	44905 672	24.3446	1.6784
5	81 669.67	1983 122.6	42837 599	24.2822	1.7195
6	78 512.82	1901 452.9	40854 477	24.2184	1.7567
7	75 486.30	1822 940.1	38953 024	24.1493	1.8131
8	72 573.55	1747 453.8	37130 084	24.0784	1.8665
9	69 773.19	1674 880.2	35382 630	24.0046	1.9226
10	67 082.89	1605 107.0	33707 750	23.9272	1.9858
11	64 494.40	1538 024.1	32102 643	23.8474	2.0481
12	62 007.02	1473 529.7	30564 619	23.7639	2.1161
13	59 610.21	1411 522.7	29091 089	23.6792	2.1763
14	57 310.63	1351 912.5	27679 566	23.5892	2.2492
15	55 095.91	1294 601.9	26327 654	23.4972	2.3178
16	52 964.65	1239 506.0	25033 052	23.4025	2.3855
17	50 913.80	1186 541.3	23793 546	23.3049	2.4527
18	48 933.55	1135 627.5	22607 005	23.2075	2.5053
19	47 019.02	1086 694.0	21471 377	23.1118	2.5375
20	45 173.07	1039 674.9	20384 683	23.0154	2.5587
21	43 399.16	994 501.9	19345 008	22.9152	2.5813
22	41 689.48	951 102.7	18350 506	22.8140	2.5939
23	40 057.18	909 413.2	17399 403	22.7029	2.6317
24	38 488.79	869 356.1	16489 990	22.5873	2.6726
25	36 975.14	830 867.3	15620 634	22.4710	2.7011
26	35 523.51	793 892.1	14789 767	22.3484	2.7384
27	34 122.38	758 368.6	13995 875	22.2250	2.7631
28	32 782.09	724 246.2	13237 506	22.0927	2.8046
29	31 489.72	691 464.1	12513 260	21.9584	2.8377
30	30 248.60	659 974.4	11821 796	21.8183	2.8749
31	29 052.91	629 725.8	11161 821	21.6751	2.9065
32	27 901.69	600 672.9	10532 096	21.5282	2.9341
33	26 796.89	572 771.2	9931 423	21.3745	2.9671
34	25 733.26	545 974.3	9358 652	21.2167	2.9967
35	24 718.28	520 241.1	8812 677	21.0468	3.0477
36	23 743.81	495 522.8	8292 436	20.8696	3.1045
37	22 804.56	471 779.0	7796 913	20.6879	3.1571
38	21 903.56	448 974.4	7325 134	20.4978	3.2174
39	21 036.26	427 070.8	6876 160	20.3016	3.2775
40	20 198.45	406 034.6	6449 089	20.1023	3.3291
41	19 392.85	385 836.1	6043 055	19.8958	3.3826
42	18 620.49	366 443.3	5657 219	19.6796	3.4445
43	17 877.10	347 822.8	5290 775	19.4563	3.5069
44	17 156.00	329 945.7	4942 952	19.2321	3.5545
45	16 463.00	312 789.7	4613 007	18.9996	3.6049
46	15 800.36	296 326.7	4300 217	18.7544	3.6678
47	15 156.35	280 526.3	4003 890	18.5088	3.7146
48	14 538.72	265 370.0	3723 364	18.2526	3.7679
49	13 944.59	250 831.2	3457 994	17.9877	3.8231
50	13 369.25	236 886.7	3207 163	17.7188	3.8699
51	12 816.48	223 517.4	2970 276	17.4398	3.9204
52	12 280.40	210 700.9	2746 759	17.1575	3.9612
53	11 768.52	198 420.5	2536 058	16.8603	4.0141
54	11 273.57	186 652.0	2337 637	16.5566	4.0629

[x]	D_x	N_x	S_x	$\ddot{a}_x$	$Var(\Sigma v^y)$
55	10 791.84	175 378.4	2150 985	16.2510	4.0996
56	10 331.94	164 586.6	1975 607	15.9299	4.1476
57	9 887.67	154 254.7	1811 020	15.6007	4.1940
58	9 453.09	144 367.0	1656 766	15.2719	4.2254
59	9 035.24	134 913.9	1512 399	14.9320	4.2599
60	8 630.31	125 878.7	1377 485	14.5857	4.2902
61	8 231.65	117 248.4	1251 606	14.2436	4.3012
62	7 852.36	109 016.7	1134 358	13.8833	4.3245
63	7 484.06	101 164.3	1025 341	13.5173	4.3433
64	7 123.74	93 680.3	924 177	13.1504	4.3516
65	6 769.82	86 556.5	830 496	12.7857	4.3461
66	6 430.74	79 786.7	743 940	12.4071	4.3463
67	6 101.97	73 356.0	664 153	12.0217	4.3444
68	5 780.68	67 254.0	590 797	11.6343	4.3357
69	5 462.79	61 473.3	523 543	11.2531	4.3122
70	5 153.99	56 010.5	462 070	10.8674	4.2852
71	4 847.43	50 856.6	406 059	10.4915	4.2416
72	4 555.04	46 009.1	355 203	10.1007	4.2050
73	4 266.10	41 454.1	309 194	9.7171	4.1565
74	3 981.18	37 188.0	267 739	9.3410	4.0966
75	3 706.13	33 206.8	230 551	8.9600	4.0360
76	3 438.40	29 500.7	197 345	8.5798	3.9716
77	3 173.94	26 062.3	167 844	8.2113	3.8959
78	2 919.93	22 888.3	141 782	7.8387	3.8215
79	2 668.06	19 968.4	118 893	7.4842	3.7342
80	2 424.65	17 300.4	98 925	7.1352	3.6429
81	2 195.62	14 875.7	81 625	6.7752	3.5596
82	1 968.01	12 680.1	66 749	6.4431	3.4633
83	1 749.47	10 712.1	54 069	6.1231	3.3631
84	1 539.78	8 962.6	43 357	5.8207	3.2568
85	1 343.62	7 422.8	34 394	5.5245	3.1507
86	1 161.69	6 079.2	26 971	5.2331	3.0473
87	993.57	4 917.5	20 892	4.9493	2.9476
88	836.14	3 923.9	15 975	4.6929	2.8450
89	688.40	3 087.8	12 051	4.4855	2.7281
90	561.43	2 399.4	8 963	4.2738	2.6123
91	453.97	1 838.0	6 563	4.0486	2.5040
92	358.74	1 384.0	4 725	3.8579	2.3878
93	280.97	1 025.3	3 341	3.6490	2.2796
94	215.24	744.3	2 316	3.4579	2.1719
95	162.82	529.0	1 572	3.2492	2.0784
96	118.49	366.2	1 043	3.0907	1.9849
97	84.16	247.7	677	2.9437	1.8951
98	58.26	163.6	429	2.8077	1.8089
99	39.26	105.3	265	2.6821	1.7264
100	25.74	66.0	160	2.5664	1.6470
101	16.39	40.3	94	2.4596	1.5701
102	10.13	23.9	54	2.3609	1.4944
103	6.08	13.8	30	2.2688	1.4178
104	3.54	7.7	16	2.1814	1.3365
105	1.99	4.2	8	2.0952	1.2443
106	1.09	2.2	4	2.0036	1.1314
107	.58	1.1	2	1.8940	.9817
108	.30	.5	1	1.7398	.7716
109	.15	.2		1.4840	.4715
110+	.07	.1		1.0000	.0000

APPENDIX B:

Mortality Commutation Functions @ 4%

[x]	C_x	M_x	R_x	A_x	$Var(v^T)$
0	502.8846	5 999.14	363 310	.05999	.09007
1	45.9860	5 496.25	357 310	.05746	.06461
2	17.6781	5 450.27	351 814	.05929	.06384
3	23.7927	5 432.59	346 364	.06147	.06506
4	13.0692	5 408.79	340 931	.06367	.06583
5	15.7057	5 395.73	335 523	.06607	.06744
6	6.7944	5 380.02	330 127	.06852	.06890
7	9.4358	5 373.23	324 747	.07118	.07111
8	9.0717	5 363.79	319 374	.07391	.07320
9	6.7090	5 354.72	314 010	.07674	.07541
10	8.3854	5 348.01	308 655	.07972	.07788
11	6.8215	5 339.62	303 307	.08279	.08033
12	11.9244	5 332.80	297 967	.08600	.08299
13	6.8781	5 320.88	292 635	.08926	.08536
14	10.4702	5 314.00	287 314	.09272	.08822
15	12.1847	5 303.53	282 000	.09626	.09090
16	13.7504	5 291.34	276 696	.09990	.09356
17	22.0300	5 277.59	271 405	.10366	.09620
18	32.4655	5 255.56	266 127	.10740	.09826
19	37.5248	5 223.10	260 872	.11108	.09952
20	36.4859	5 185.57	255 649	.11479	.10036
21	40.4781	5 149.09	250 463	.11864	.10124
22	28.8620	5 108.61	245 314	.12254	.10173
23	27.7319	5 079.75	240 205	.12681	.10322
24	33.3076	5 052.02	235 126	.13126	.10482
25	29.5090	5 018.71	230 074	.13573	.10594
26	34.8404	4 989.20	225 055	.14045	.10740
27	27.8885	4 954.36	220 066	.14519	.10837
28	31.5212	4 926.47	215 111	.15028	.11000
29	29.9758	4 894.95	210 185	.15545	.11130
30	32.2846	4 864.97	205 290	.16083	.11275
31	33.8019	4 832.69	200 425	.16634	.11400
32	31.6577	4 798.89	195 592	.17199	.11508
33	32.9808	4 767.23	190 793	.17790	.11637
34	25.2384	4 734.25	186 026	.18397	.11753
35	23.7676	4 709.01	181 292	.19051	.11953
36	26.0269	4 685.24	176 583	.19732	.12176
37	23.9009	4 659.22	171 898	.20431	.12382
38	24.8521	4 635.31	167 238	.21162	.12619
39	28.7226	4 610.46	162 603	.21917	.12855
40	28.7440	4 581.74	157 993	.22684	.13057
41	26.4787	4 553.00	153 411	.23478	.13267
42	27.2146	4 526.52	148 858	.24309	.13509
43	33.5196	4 499.30	144 331	.25168	.13754
44	33.1573	4 465.78	139 832	.26030	.13941
45	29.4434	4 432.63	135 366	.26925	.14139
46	36.3104	4 403.18	130 934	.27868	.14385
47	34.6847	4 366.87	126 531	.28812	.14569
48	34.9489	4 332.19	122 164	.29798	.14778
49	39.0180	4 297.24	117 831	.30817	.14994
50	38.5651	4 258.22	113 534	.31851	.15178
51	43.1324	4 219.66	109 276	.32924	.15376
52	39.5571	4 176.52	105 056	.34010	.15536
53	42.3214	4 136.97	100 880	.35153	.15744
54	48.1295	4 094.64	96 743	.36321	.15935

[x]	C_x	M_x	R_x	A_x	$Var(v^T)$
55	44.8276	4 046.51	92 648	.37496	.16079
56	46.8911	4 001.69	88 602	.38731	.16267
57	54.2871	3 954.80	84 600	.39997	.16449
58	54.2643	3 900.51	80 645	.41262	.16572
59	57.4259	3 846.24	76 745	.42569	.16707
60	66.7189	3 788.82	72 898	.43901	.16826
61	62.6872	3 722.10	69 110	.45217	.16870
62	66.2921	3 659.41	65 388	.46603	.16961
63	72.4658	3 593.12	61 728	.48010	.17035
64	79.9366	3 520.65	58 135	.49421	.17067
65	78.6991	3 440.72	54 614	.50824	.17046
66	81.4354	3 362.02	51 174	.52280	.17046
67	86.6010	3 280.58	47 812	.53763	.17039
68	95.5479	3 193.98	44 531	.55253	.17005
69	98.6980	3 098.43	41 337	.56719	.16913
70	108.3329	2 999.74	38 239	.58202	.16807
71	105.9442	2 891.40	35 239	.59648	.16636
72	113.7446	2 785.46	32 347	.61151	.16492
73	120.8456	2 671.72	29 562	.62627	.16302
74	121.9235	2 550.87	26 890	.64073	.16067
75	125.1888	2 428.95	24 339	.65539	.15830
76	132.2130	2 303.76	21 911	.67001	.15577
77	131.9321	2 171.54	19 607	.68418	.15280
78	139.5672	2 039.61	17 435	.69851	.14988
79	140.7915	1 900.04	15 396	.71214	.14646
80	135.7805	1 759.25	13 496	.72557	.14288
81	143.1583	1 623.47	11 736	.73942	.13961
82	142.8510	1 480.31	10 113	.75219	.13583
83	142.3965	1 337.46	8 633	.76450	.13190
84	136.9370	1 195.07	7 295	.77613	.12773
85	130.2539	1 058.13	6 100	.78752	.12357
86	123.4409	927.88	5 042	.79873	.11952
87	119.2189	804.44	4 114	.80964	.11561
88	115.5799	685.22	3 310	.81950	.11158
89	100.4929	569.64	2 624	.82748	.10700
90	85.8607	469.14	2 055	.83562	.10246
91	77.7693	383.28	1 586	.84428	.09821
92	63.9738	305.51	1 202	.85162	.09365
93	54.9221	241.54	897	.85966	.08941
94	44.1457	186.62	655	.86701	.08518
95	38.0654	142.47	469	.87503	.08152
96	29.7780	104.41	326	.88113	.07785
97	22.6608	74.63	222	.88678	.07433
98	16.7534	51.97	147	.89201	.07095
99	12.0185	35.21	95	.89684	.06771
100	8.3566	23.20	60	.90129	.06460
101	5.6264	14.84	37	.90540	.06158
102	3.6653	9.21	22	.90920	.05861
103	2.3089	5.55	13	.91274	.05561
104	1.4058	3.24	7	.91610	.05242
105	.8272	1.83	4	.91942	.04880
106	.4703	1.01	2	.92294	.04437
107	.2585	.54	1	.92715	.03850
108	.1374	.28		.93308	.03026
109	.0706	.14		.94292	.01849
110+	.0688	.07		.96154	.00000

APPENDIX C:

Mean of log-normal lifetime at birth

$$f_{T_x}(y) = {}_y p_x \mu_{x+y} = \frac{\bar{f}_{T_0}(x+y)}{S_{T_0}(x)} = \frac{\phi([\log(\omega-x-y)-\mu]/\sigma)}{\sigma(\omega-x-y)\Phi([\log(\omega-x)-\mu]/\sigma)}$$

$$Mean(T_x) = \int_0^{\omega-x} y f_{T_x}(y) dy$$

$$\begin{aligned} u &= [\log(\omega-x-y)-\mu]/\sigma \\ \mu + \sigma u &= \log(\omega-x-y) \\ \exp(\mu + u\sigma) &= \omega-x-y \\ y &= \omega-x-\exp(\mu+u\sigma) \\ du &= 1/\sigma(\omega-x-y)dy \end{aligned}$$

$$\begin{aligned} &= \int_{-\infty}^{[\log(\omega-x)-\mu]/\sigma} \omega-x-\exp(\mu+u\sigma) f_{T_x}(y) dy \\ &= \omega-x-e^{\mu} \int_{-\infty}^{[\log(\omega-x)-\mu]/\sigma} \exp(u\sigma) \frac{\phi(u)}{S_{T_0}(x)} du \\ &= \omega-x-e^{\mu} \int_{-\infty}^{[\log(\omega-x)-\mu]/\sigma} \exp(u\sigma) \frac{1}{S_{T_0}(x)\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \end{aligned}$$

$$\frac{1}{2}u^2 - u\sigma = \frac{1}{2}(u^2 - 2u\sigma + \sigma^2 - \sigma^2) = \frac{1}{2}(u-\sigma)^2 - \frac{1}{2}\sigma^2$$

$$\begin{aligned} &= \omega-x-e^{\mu} \int_{-\infty}^{[\log(\omega-x)-\mu]/\sigma} \frac{1}{S_{T_0}(x)\sqrt{2\pi}} e^{-\frac{1}{2}(u-\sigma)^2 + \frac{1}{2}\sigma^2} du \\ &= \omega-x - \frac{e^{\mu+\frac{1}{2}\sigma^2}}{S_{T_0}(x)} \int_{-\infty}^{[\log(\omega-x)-\mu]/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(u-\sigma)^2} du \\ &= \omega-x - \frac{e^{\mu+\frac{1}{2}\sigma^2}}{S_{T_0}(x)} \int_{-\infty}^{[\log(\omega-x)-\mu]/\sigma-\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \\ &= \omega-x - e^{\mu+\frac{1}{2}\sigma^2} \frac{\Phi([\log(\omega-x)-\mu]/\sigma-\sigma)}{\Phi([\log(\omega-x)-\mu]/\sigma)} \end{aligned}$$